HW 4 Solution. 2270-2

Sec. 3.3.

#8. $\begin{bmatrix}1&-3\\2&-6\\3&-9\end{bmatrix} \to \begin{bmatrix}1&-3\\0&0\\0&0\end{bmatrix} \Rightarrow \left.\begin{matrix}x_1=3x_2\\x_2=x_2\end{matrix}\right\} \Rightarrow \begin{bmatrix}x_1\\x_2\end{bmatrix}=t\begin{bmatrix}3\\1\end{bmatrix}$

So $\begin{bmatrix}-3\\-6\\-9\end{bmatrix}$ is redundant, since $\begin{bmatrix}-3\\-6\\-9\end{bmatrix}=-3\begin{bmatrix}1\\2\\3\end{bmatrix}$ & a basis of $\ker(A)=\begin{bmatrix}3\\1\end{bmatrix}$,

and the image (A) has a basis, $\begin{bmatrix}1\\2\\3\end{bmatrix}$

#10. $\begin{bmatrix}0&1&1\\0&1&2\\0&1&3\end{bmatrix} \to \begin{bmatrix}0&1&1\\0&0&1\\0&0&2\end{bmatrix} \to \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix} \Rightarrow \text{"}\begin{pmatrix}x_1=x_1\\x_2=0\\x_3=0\end{pmatrix} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=t\begin{bmatrix}1\\0\\0\end{bmatrix}$.

So $\begin{bmatrix}0\\0\\0\end{bmatrix}$ is redundant & a basis of $\ker(A)$ is $\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$ and a basis of $\mathrm{Im}(A)$

are 2nd & 3rd columns $\left(\begin{bmatrix}1\\1\\1\end{bmatrix} \& \begin{bmatrix}1\\2\\3\end{bmatrix}\right)$.

#18. $\begin{bmatrix}1&-2&0&-1&0\\0&0&1&5&0\\0&0&0&0&1\end{bmatrix} \Rightarrow \begin{matrix}x_1=2x_2+x_4, & x_3=-5x_4\\x_2=x_2, & x_4=x_4\\ & x_5=0\end{matrix} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=t\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}1\\0\\-5\\1\\0\end{bmatrix}$.

$\begin{bmatrix}-2\\0\\0\end{bmatrix}=-2\begin{bmatrix}1\\0\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\5\\0\end{bmatrix}=-\begin{bmatrix}1\\0\\0\end{bmatrix}+5\begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}-2\\0\\0\end{bmatrix}$ & $\begin{bmatrix}-1\\5\\0\end{bmatrix}$ are redundant.

& A basis of $\ker(A)=\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix},\begin{bmatrix}1\\0\\-5\\1\\0\end{bmatrix}$ & a basis of $\mathrm{Im}(A)=\left(\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}0\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$.

#24. $\begin{bmatrix}4&8&1&1&6\\3&6&1&2&5\\2&4&1&9&10\\1&2&3&2&0\end{bmatrix} \to \begin{bmatrix}1&2&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix} \Rightarrow$ 2nd column $\begin{bmatrix}8\\6\\4\\2\end{bmatrix}$ is redundant &

$\begin{matrix}x_1=-2x_2\\x_2=x_2\\x_3=0\\x_4=0\\x_5=0\end{matrix} \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=t\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix} \Rightarrow$ A basis of $\ker(A)=\left(\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}\right)$.

& A basis of $\mathrm{Im}(A)=$ 1st, 3rd, 4th & 5th columns.
$=\left(\begin{bmatrix}4\\3\\2\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\\3\end{bmatrix},\begin{bmatrix}1\\2\\9\\2\end{bmatrix},\begin{bmatrix}6\\5\\10\\0\end{bmatrix}\right)$.

#26. $\left\{\begin{matrix}x_1=-\frac{3}{2}x_2-\frac{1}{2}x_3\\x_2=x_2\\x_3=x_3\end{matrix}\right. \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=t\begin{bmatrix}-\frac{3}{2}\\1\\0\end{bmatrix}+s\begin{bmatrix}-\frac{1}{2}\\0\\1\end{bmatrix}. \Rightarrow$ a basis is $\left(\begin{bmatrix}-\frac{3}{2}\\1\\0\end{bmatrix},\begin{bmatrix}-\frac{1}{2}\\0\\1\end{bmatrix}\right)$.

"linearly independent"

#28. They must be linearly independent, since $\mathbb{R}^4$ has dim 4.

$\begin{bmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\2&3&4&k\end{bmatrix} \to \begin{bmatrix}1&0&0&2\\0&1&0&3\\0&0&1&4\\0&0&0&k-29\end{bmatrix} \Rightarrow$ if $k-29\neq0$, $\mathrm{rref}(A)=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$ so

they are linearly indep. $\Rightarrow \boxed{k\neq29}$.

#30. $\left\{\begin{matrix}x_1=\frac{1}{2}x_2-x_3-2x_4\\x_2=x_2\\x_3=x_3\\x_4=x_4\end{matrix}\right. \Rightarrow \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=t\begin{bmatrix}\frac{1}{2}\\1\\0\\0\end{bmatrix}+s\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+r\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}$.

are linearly indep.

A basis of this space is $\left(\begin{bmatrix}\frac{1}{2}\\1\\0\\0\end{bmatrix},\begin{bmatrix}-1\\0\\1\\0\end{bmatrix},\begin{bmatrix}-2\\0\\0\\1\end{bmatrix}\right)$.

Sec. 3.3 (continued)

#32. Find $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ s.t. $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} = 0$ & $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = 0$. $\Rightarrow \begin{pmatrix} x_1 - x_3 + x_4 = 0 \\ x_2 + 2x_3 + 3x_4 = 0 \end{pmatrix}$

dot product

$\Rightarrow A = \begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ & a basis of $\ker(A)$ is the answer.
Since x_3 & x_4 are free, a basis is $\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ //

#36. No! By the nullity-rank thm, $3 = \dim(\text{im}(A)) + \dim(\ker(A))$.
If $\text{im}(A) = \ker(A)$, then $\dim(\text{im}(A)) = \dim(\ker(A)) = a$ "say"
$\Rightarrow 3 = a + a \Rightarrow 2a = 3 \Rightarrow a = \frac{3}{2}$ which is impossible as dimension & not an integer. //

#42. Let $\vec{v}_1, \ldots, \vec{v}_n$ be a basis of $V \subseteq \mathbb{R}^n$.
Then consider them as n linearly indep. vectors in $\mathbb{R}^n$.
We know $\dim(\mathbb{R}^n) = n$ as well. Then by Fact 3.3.4 (c) (p.126), they form a basis of $\mathbb{R}^n$. $\Rightarrow V = \text{Span}(\vec{v}_1, \ldots, \vec{v}_n) = \mathbb{R}^n \Rightarrow V = \mathbb{R}^n$. //

Sec 3.4

#4. Solve $\begin{bmatrix} 23 \\ 29 \end{bmatrix} = c_1 \begin{bmatrix} 46 \\ 58 \end{bmatrix} + c_2 \begin{bmatrix} 61 \\ 67 \end{bmatrix}$. $\Rightarrow \left[\begin{array}{cc|c} 46 & 61 & 23 \\ 58 & 67 & 29 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{array}\right]$.

So $\begin{bmatrix} 23 \\ 29 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 46 \\ 58 \end{bmatrix}$ So it is in the span of $\vec{v}_1$ & $\vec{v}_2$.

#10. $\left[\begin{array}{cc|c} -1 & -2 & -5 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 5 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} -5 \\ 1 \\ 3 \end{bmatrix} = 3\vec{v}_1 + 1 \cdot \vec{v}_2$.
So it is in the span of $\vec{v}_1$ & $\vec{v}_2$.

#12. $\left[\begin{array}{cc|c} 8 & 5 & 1 \\ 4 & 2 & -2 \\ -1 & -1 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 8 & 5 & 1 \\ 4 & 2 & -2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -3 & -15 \\ 0 & -2 & -10 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right]$

$\Rightarrow \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} = -3\vec{v}_1 + 5\vec{v}_2$. So it is in the span of $\vec{v}_1$ & $\vec{v}_2$.

#16. $\left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 3 & 1 \\ 1 & 3 & 6 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 2 & -6 \\ 0 & 2 & 5 & -4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 13 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & 1 & 8 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 21 \\ 0 & 1 & 0 & -22 \\ 0 & 0 & 1 & 8 \end{array}\right]$.

$\Rightarrow \begin{bmatrix} 7 \\ 1 \\ 8 \end{bmatrix} = 21\vec{v}_1 + (-22)\vec{v}_2 + 8\vec{v}_3$. So it is in the span of $\vec{v}_1, \vec{v}_2$ & $\vec{v}_3$.

#26. $B = [\,[A\vec{v}_1]_{\mathfrak{B}}, [A\vec{v}_2]_{\mathfrak{B}}\,]$ & $B = S^{-1}AS$ where $S = [\vec{v}_1, \vec{v}_2]$.

$\Rightarrow B = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = -\frac{1}{1} \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 8 & 5 \end{bmatrix} = -\begin{bmatrix} -5 & -4 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ -4 & -3 \end{bmatrix}$.

Sec 3.4 (continued)

#44. $\vec{x} = (\mathfrak{B}\text{-matrix}) \cdot [\vec{x}]_{\mathfrak{B}}$ or if $\vec{v_1}$ & $\vec{v_2}$ are basis of V,
then $\vec{x} = [\vec{v_1}\ \vec{v_2}] \cdot [\vec{x}]_{\mathfrak{B}}$.

$$\Rightarrow \quad \vec{x} = \begin{bmatrix} 8 & 5 \\ 4 & 2 \\ -1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 11 \\ 6 \\ -1 \end{bmatrix} \text{ //}$$

#46. Choose a vector $\vec{v_1}$ in the plane which is not a multiple of $\vec{x}$. For example, let $\vec{v_1} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$.

Then, since $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{x} = 2\vec{v_1} - \vec{v_2}$, for $\mathfrak{B} = (\vec{v_1}, \vec{v_2})$.

So $\vec{v_2} = 2\vec{v_1} - \vec{x} = 2\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4-1 \\ -2+1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$.

Then $(\vec{v_1}, \vec{v_2}) = \left(\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \right)$ is a basis of V

which has $[\vec{x}]_{\mathfrak{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

#62. the standard matrix $= \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} = A$ & $\mathfrak{B}$-matrix $= \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = B$.

where $\mathfrak{B} = (\vec{v_1}, \vec{v_2})$ & $S = [\vec{v_1}, \vec{v_2}] \overset{\text{let}}{=} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ //

Then use $AS = SB$

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} a+2c & b+2d \\ 4a+3c & 4b+3d \end{bmatrix} = \begin{bmatrix} 5a & -b \\ 5c & -d \end{bmatrix}.$$

$$\Rightarrow \begin{cases} a+2c = 5a \\ 4a+3c = 5c \end{cases} \ \& \ \begin{cases} b+2d = -b \\ 4b+3d = -d \end{cases} \Rightarrow \begin{cases} c = 2a \\ d = -b. \end{cases}$$

$$\Rightarrow S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2a & -b \end{bmatrix}, \quad \text{let } a=1 \ \& \ b=1.$$

$$\Rightarrow S = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \Rightarrow \mathfrak{B} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \text{ //}$$