Mathematics 1180
MATHEMATICS FOR LIFE SCIENTISTS
Computer Assignment III
Due January 28, 2002

We will compare the solution of a one dimensional equation describing a disease,

$$\frac{di}{dt} = \alpha i(1-i) - \mu i, \tag{1}$$

with a system of differential equations describing the same and related processes. i is the fraction of sick people, α the infection rate and μ the removal rate. We can use the usual series of commands involving `dsolve` to find and plot the solution (remember to give your solution a name other than i).

To follow the dynamics of the actual numbers of infected and susceptible people, we can study a **system** of differential equations for i and s (the number of susceptible people)

$$\begin{aligned} \frac{di}{dt} &= \alpha is - \mu i \\ \frac{ds}{dt} &= -\alpha is + \mu i. \end{aligned} \tag{2}$$

To plot the solution of this system, we will use a new command `DEplot` as follows:

```
> with(DEtools):
> diffsys := [diff(i(t),t)=alpha*i(t)*s(t)-mu*i(t),
              diff(s(t),t)= -alpha*i(t)*s(t)+mu*i(t)];
> DEplot(diffsys,[i(t),s(t)],t=0..6,{[0,i0,1-i0]},
          i=0..1,s=0..1,scene=[i(t),s(t)],stepsize=0.1,linecolor=blue);
```

The first line tells Maple to use the `DEtools` package. The next line defines the differential equation in the form demanded by `DEplot`. Be careful about the various types of brackets and parentheses. The `DEplot` command requires at least 4 arguments, the differential equation, a list of variables, the time range, and the initial conditions. Initial conditions are listed in the same order as the variables: `[0,i_0,s_0]` tells Maple to start from $(t = 0, i = i_0, s = s_0)$ and must appear inside curly brackets (and must include an initial condition for t).

`DEplot` can take numerous optional arguments. Included here are limits for the variables i and s (both can run from 0 to 1), and `scene`, which tells Maple to put t on the x-axis and i on the y-axis. The step size option, `stepsize=0.1`, forces Maple to use a smaller step size in its solving routine (a fancy version of Euler's method). Without this command, Maple breaks the range into 6 pieces and gives a pretty pathetic graph; by setting the step size to 0.1, we break the region from $t = 0$ to $t = 6$ into 60 pieces and get a nice graph. The linecolor option produces a color that prints up better.

PROBLEMS

- **1.** Set $\alpha = 8$, $\mu = 6$ and $i_0 = 0.01$ (the initial condition). Graph the solution of equation 1 for $0 \le t \le 6$. Describe in words what is happening. What is the fraction of infected people at $t = 0$, $t = 1$, $t = 2$, $t = 3$ and $t = 5$? Mark these points on your graph.

- **2.** We now compare the solution of equation 1 with the two dimensional system equation 2 using the same parameter values.

  a. Use `DEplot` to produce a graph of i as a function of time for equation 2. Use $s(0) = s_0 = 0.99$ as your initial condition for s (can you see why?). How does the solution compare with **1**?

  b. Modify the `scene` command to produce a graph of s as a function of time. How is s related to i? Can you see why?

  c. Modify the `scene` command to produce a graph of s as a function of i for equation 2 (a solution in the phase-plane). From the graphs of i and s as functions of t, mark where $t = 0$, $t = 1$, $t = 2$, $t = 3$ and $t = 5$ occur in the phase-plane.

- **3.** Now consider the following modification of the equations.

$$\begin{aligned} \frac{di}{dt} &= \alpha i s - \mu i \\ \frac{ds}{dt} &= -\alpha i s + \beta s, \end{aligned} \tag{3}$$

  The α term is the same as above, but μ now represents death due to the disease. The β term describes births of new susceptible offspring from susceptible parents.

  a. Convince yourself and other relevant people that equation 3 is identical to the predator-prey equations in the book. Which is the predator?

  b. Set $\beta = 1$ and follow the steps in **2.** Can you explain why the dynamics are so completely different from **2**?

  c. Draw the nullclines on your figure (unless you are cleverer than us, you'll have to do this by hand).

  d. Try the same steps as in **b** with $\beta = 0.1$. Describe what happens. Does it remind you of anything? What would happen if β were made very small?