Homework 3 Solutions

Sec. 2.5 ① $y' = -y$ interval $[0, 0.5]$

$y(0) = 2$ $h = 0.1$

$x_0 = 0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4,\ x_5 = 0.5$

Improved Euler: $k_1 = f(x_n, y_n) = -y_n$

$u_{n+1} = y_n + h k_1 = y_n + 0.1(-y_n)$

$k_2 = f(x_{n+1}, u_{n+1})$

$y_{n+1} = y_n + h \frac{1}{2}(k_1 + k_2) = y_n + 0.05(k_1 + k_2)$

$y_0 = y(0) = 2$

$n = 1$: $k_1 = f(y_0) = -2$

$u_1 = 2 + 0.1(-2) = 2 - 0.2 = 1.8$

$k_2 = f(x_1, u_1) = -u_1 = -1.8$

$y_1 = 2 + 0.05(-2 - 1.8) = 1.81$

$n=2$: $k_1 = f(y_1) = -1.81$

$u_2 = 1.81 + 0.1(-1.81) = 1.629$

$k_2 = -1.629$

$y_2 = 1.81 + 0.05(-1.81 - 1.629) = 1.63805$

$n=3$: $k_1 = -1.63805$

$u_3 = 1.63805 + 0.1(-1.63805) = 1.4742$

$k_2 = -1.4742$

$y_3 = 1.63805 + 0.05[-1.63805 - 1.4742] = 1.4824$

$n=4$: $k_1 = -y_3 = -1.4824$

$u_4 = 1.4824 + 0.1(-1.4824) = 1.3342$

$k_2 = -u_4 = -1.3342$

$y_4 = 1.4824 + 0.05[-1.4824 - 1.3342] = 1.3416$

$n=5$: $k_1 = -y_4 = -1.3416$

$u_5 = 1.3416 + 0.1(-1.3416) = 1.2074$

$k_2 = -1.2074$

$y_5 = 1.3416 + 0.05(-1.3416 - 1.2074) = 1.2142$

Table:

$x =$	0	0.1	0.2	0.3	0.4	0.5
y_n	2	1.81	1.6381	1.4824	1.3342	1.2142
$y(x) =$	2	1.8097	1.6375	1.4816	1.3406	1.2131

$y(x) = 2e^{-x}$

(4) $y' = x - y$ $\quad$ $h = 0.1$ interval $[0, 0.5]$

$y(0) = 1$ $\quad$ $y(x) = 2e^{-x} + x - 1$ exact soln.

$f(x,y) = x - y$

General n:

$$k_1 = x_n - y_n = 0.n - y_n$$

$$u_{n+1} = y_n + h\overbrace{f(x_n, y_n)}^{k_1} = y_n + 0.1(0.n - y_n)$$

$$k_2 = f(x_{n+1}, u_{n+1}) = 0.(n+1) - u_{n+1}$$

$$y_{n+1} = y_n + 0.05(k_1 + k_2)$$

starting with $y_0 = 1 = y(0)$.

$n=1$: (y_1)

$$k_1 = 0 - y_0 = -1$$

$$u_1 = 1 + 0.1(0-1) = 0.9$$

$$k_2 = 0.1 - 0.9 = -0.8$$

$$y_1 = 1 + 0.05(-1 - 0.8) = 0.91$$

y_2:

$$k_1 = 0.1 - 0.91 = -0.81$$

$$u_2 = 0.91 + 0.1(-0.81) = 0.829$$

$$k_2 = 0.2 - 0.829 = -0.629$$

$$y_2 = 0.91 + 0.05(-0.81 - 0.629) = 0.8381$$

y_3:

$$k_1 = 0.2 - 0.8381 = -0.6381$$

$$u_3 = 0.8381 + 0.1(-0.6381) = 0.7743$$

$$k_2 = 0.3 - 0.7743 = -0.4743$$

$$y_3 = 0.8381 + 0.05(-0.6381 - 0.4743) = 0.7825$$

y_4: $k_1 = 0.3 - 0.7825 = -0.4825$

$u_4 = 0.7825 + 0.1(-0.4825) = 0.7343$

$k_2 = 0.4 - 0.7343 = -0.3343$

$y_4 = 0.7825 + 0.05(-0.4825 - 0.3343)$

$= 0.7417$

y_5: $k_1 = 0.4 - 0.7417 = -0.3417$

$u_5 = 0.7417 + 0.1(-0.3417) = 0.7075$

$k_2 = 0.5 - 0.7075 = -0.2075$

$y_5 = 0.7417 + 0.05(-0.3417 - 0.2075) = 0.7142$

Table:

x:	0	0.1	0.2	0.3	0.4	0.5
y_n	1	0.91	0.8381	0.7825	0.7417	0.7142
$y(x)$	1	0.9097	0.8375	0.7816	0.7406	0.7131

$y(x) = 2e^{-x} + x - 1$

Sec. 3.1 (7) $y''+y'=0, \quad y_1=1, \; y_2=e^{-x}$

$y(0)=-2$
$y'(0)=8$

$y_1'=0, \; y_1''=0$ so $y_1''+y_1'=0$

$y_2'=-e^{-x}, \; y_2''=e^{-x} \Rightarrow y_2''+y_2'=e^{-x}-e^{-x}=0$

$y=C_1y_1+C_2y_2=C_1+C_2e^{-x}$

$y'=-C_2e^{-x}$

$y(0)=C_1+C_2=-2$
$y'(0)=-C_2=8$
$\Rightarrow C_2=-8$, $C_1-8=-2$ $\Rightarrow C_1=6$, $C_2=-8$

$\boxed{y=6-8e^{-x}}$

(10) $y''-10y'+25y=0, \; y_1=e^{5x}, \; y_2=xe^{5x}$

$y(0)=3$
$y'(0)=13$

$y_1'=5e^{5x}, \; y_1''=25e^{5x}$

$y_1''-10y_1'+25y=25e^{5x}-50e^{5x}+25e^{5x}=0$

$y=C_1e^{5x}+C_2xe^{5x}$

$y'=5C_1e^{5x}+C_2e^{5x}+5C_2xe^{5x}$

$y(0)=C_1=3$
$y'(0)=5C_1+C_2=13$
$\Rightarrow C_1=3$, $C_2=-2$ $\Rightarrow \boxed{y=3e^{5x}-2xe^{5x}}$

(20) $f(x) = \pi$ (const.)

$g(x) = \cos^2 x + \sin^2 x$

Since $g(x) = 1$, $f(x) = \pi = \pi \cdot 1 = \pi g(x)$

$\Rightarrow$ f, g linearly dependant. (on $\mathbb{R}$)

(23) $f(x) = xe^x$, $g(x) = |x| e^x$

Are linearly independent on $\mathbb{R}$. If $f(x) = c\,g(x)$ for some constant c then $xe^x = c|x|e^x$

$\Rightarrow$ $x = c|x|$ $\Rightarrow$ $\dfrac{x}{|x|} = c$ all $x \neq 0$. But

$\dfrac{x}{|x|} = -1$ if $x<0$, and $\dfrac{x}{|x|} = 1$ $x>0$. Contradiction.

(27) y_p a particular solution of $y'' + py' + qy = f(x)$ and y_c a solution of $y'' + py' + qy = 0$. Then if

$y = y_c + y_p$: $(y_c+y_p)'' + p(y_c+y_p)' + q(y_c+y_p) =$

$$= y_c'' + y_p'' + p(y_c' + y_p') + qy_c + qy_p$$

$$= \underbrace{y_c'' + py_c' + qy_c}_{0} + \underbrace{y_p'' + py_p' + qy_p}_{f}$$

$$= 0 + f = f$$

and therefore $y_c + y_p$ is a soln. of the non-homog. equation

(29) $y_1 = x^2$, $y_2 = x^3$; $x^2y'' - 4xy' + 6y = 0$

$y_1' = 2x$, $y_1'' = 2$ so

$$x^2y_1'' - 4xy_1' + 6y_1 = 2x^2 - 4x\cdot 2x + 6x^2 = 8x^2 - 8x^2 = 0$$

$$y_1(0) = 0, \quad y_1'(0) = 2\cdot 0 = 0$$

$y_2' = 3x^2$, $y_2'' = 6x$ so

$$x^2y_2'' - 4xy_2' + 6y_2 = x^2\cdot 6x - 4x\cdot 3x^2 + 6\cdot x^3 = 6x^3 - 12x^3 + 6x^3 = 0$$

$$y_2(0) = 0^3 = 0, \quad y_2'(0) = 3\cdot 0 = 0.$$

This does not contradict the existence + uniqueness theorem because the differential equation is NOT in standard form. (x^2 coefficient of y''). If put in standard form by dividing by x^2, the coefficients of y', y become discontinuous at $x=0$.

(34) $y'' + 2y' - 15y = 0$. Characteristic equation is

$r^2 + 2r - 15 = 0 \Rightarrow (r+5)(r-3) = 0$ roots: $r = -5, 3$

so ~~general~~ distinct roots $\Rightarrow$ e^{-5x}, e^{3x} are lin. ind.

Solutions $\Rightarrow$ $\boxed{y = c_1e^{-5x} + c_2e^{3x}}$ general soln.

(35) $y'' + 5y' = 0$. Characteristic equation is

$$r^2 + 5r = 0 \Rightarrow r(r+5) = 0 \quad r = 0, -5$$

$\Rightarrow e^{0x}, e^{-5x}$ or $1, e^{-5x}$ are lin. ind. soln.'s

$\Rightarrow$ $\boxed{y = c_1 + c_2 e^{-5x} \text{ is the general solution}}$

(51) $ax^2 y'' + bxy' + cy = 0$

a) Suppose $x > 0$. Let $v = \ln x$.

$$y' = \frac{dy}{dx} = \frac{dy}{dv}\frac{dv}{dx} = \frac{1}{x}\frac{dy}{dv}$$

$$y'' = \frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d}{dx}\frac{dy}{dv} = \boxed{\cancel{-\frac{1}{x^2}\frac{dy}{dv}\,\frac{d^2y}{dv^2}\frac{dv}{dx}}}$$

$$= -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x}\frac{d^2y}{dv^2}\frac{dv}{dx} = -\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}$$

Substituting into the ODE:

$$ax^2\left(-\frac{1}{x^2}\frac{dy}{dv} + \frac{1}{x^2}\frac{d^2y}{dv^2}\right) + bx\frac{1}{x}\frac{dy}{dv} + cy = 0$$

$$a\frac{d^2y}{dv^2} - a\frac{dy}{dv} + b\frac{dy}{dv} + cy = 0$$

$$\boxed{a\frac{d^2y}{dv^2} + (b-a)\frac{dy}{dv} + cy = 0}$$

b. The characteristic roots of this const. coeff. equation satisfy $ar^2 + (b-a)r + c = 0 \Rightarrow$

$$r = \frac{-(b-a)}{2a} \pm \frac{\sqrt{(b-a)^2 - 4ac}}{2a}.$$ Call those r_1, r_2.

If real + distinct then the solution is (general)

$$y(v) = c_1 e^{r_1 v} + c_2 e^{r_2 v}$$

but $v = \ln x$ so

$$y(x) = c_1 e^{r_1 \ln x} + c_2 e^{r_2 \ln x}$$

$$\Rightarrow y(x) = c_1 x^{r_1} + c_2 x^{r_2}.$$

Sec. 3.2 (13) $y^{(3)} + 2y'' - y' - 2y = 0$

$y(0) = 1$, $y'(0) = 2$, $y''(0) = 0$

$y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{-2x}$. So the solution of the I.V.P. is of form.

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$$

We differentiate twice + plug in the initial conditions to solve for c_1, c_2, c_3:

$$y' = c_1 e^x - c_2 e^{-x} - 2c_3 e^{-2x}$$

$$y'' = c_1 e^x + c_2 e^{-x} + 4c_3 e^{-2x}$$

$$y(0) = c_1 + c_2 + c_3 = 1$$

$$y'(0) = c_1 - c_2 - 2c_3 = 2$$

$$y''(0) = c_1 + c_2 + 4c_3 = 0$$

Solve $\rightarrow$ $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & -1 & -2 & 2 \\ 1 & 1 & 4 & 0 \end{array}\right)$

Gauss Jordan $\Rightarrow$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -2 & -3 & 1 \\ 0 & 0 & 3 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 4/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/3 \end{array}\right)$$

$c_1 = 4/3$, $c_2 = 0$, $c_3 = -1/3$

$$\boxed{y(x) = \frac{4}{3}e^{x} - \frac{1}{3}e^{-2x}}$$

(24) $y'' - 2y' + 2y = 2x$

$y(0) = 4$, $y'(0) = 8$

$y_c = c_1 e^{x} \cos x + c_2 e^{x} \sin x$, $y_p = x+1$.

The solution will have the form $y = y_c + y_p$ $\Rightarrow$

$y = c_1 e^{x} \cos x + c_2 e^{x} \sin x + x+1$

We differentiate once, plug in initial conditions + solve for x:

$y' = c_1 e^{x} \cos x - c_1 e^{x} \sin x + c_2 e^{x} \sin x + c_2 e^{x} \cos x + 1$

$y(0) = c_1 + 1 = 4$ $\Rightarrow$ $c_1 = 3$

$y'(0) = c_1 + c_2 + 1 = 8$ $\quad c_2 = 8 - 3 - 1 = 4$

$$\boxed{y = 3e^{x} \cos x + 4e^{x} \sin x + x+1}$$

(36) $y'' + p(x)y' + q(x)y = 0$ p, q continuous on I

Assume $y_1(x)$ is a known solution.

Set $y_2(x) = v(x)y_1(x)$ for some $v(x)$ and

for y_2 to be a soln, we have:

$$y_2' = v'y_1 + vy_1'$$

$$y_2'' = v''y_1 + v'y_1' + v'y_1' + vy_1'' = v''y_1 + 2v'y_1' + vy_1''$$

Now

$$y_2'' + py_2' + gy_2 = 0 \iff$$

$$y_1v'' + 2v'y_1' + vy_1'' + p(v'y_1 + vy_1') + gvy_1 = 0 \iff$$

$$y_1v'' + 2v'y_1' + pv'y_1 + vy_1'' + vpy_1' + vgy_1 = 0$$

$$y_1v'' + (2y_1' + p)v' + v\underbrace{(\cancel{y_1'' + py_1' + gy_1})}_{\substack{\parallel \\ 0}} = 0$$

since y_1 a soln.

$\iff$ $y_1v'' + (2y_1' + p)v' = 0$. A separable, 1st equation in v'!

Sec. 3.3

(3) $y'' + 3y' - 10y = 0$

Characteristic equation is $r^2 + 3r - 10 = 0$, $(r-2)(r+5) = 0$

roots: $r = 2, -5$. So general solution is $\boxed{y(x) = C_1e^{-5x} + C_2e^{2x}}$

(18) $y^{(4)} = 16y \Rightarrow y^{(4)} - 16y = 0$ Characteristic equation

is $r^4 - 16 = 0 \Rightarrow (r^2-4)(r^2+4) = 0$

roots! $\begin{matrix} r^2 = 4 \\ r^2 = -4 \end{matrix} \Rightarrow \begin{matrix} r = -2, 2 \\ r = 2i, -2i \end{matrix} \Rightarrow$ indep. soln.'s e^{-2x}, e^{2x}, $\cos 2x, \sin 2x$

$\Rightarrow$ general solution is

$$y(x) = c_1 e^{-2x} + c_2 e^{2x} + c_3 \cos 2x + c_4 \sin 2x$$

(21) $y'' - 4y' + 3y = 0$

$y(0) = 7$

$y'(0) = 11$

Characteristic eq: $r^2 - 4r + 3 = 0 \Rightarrow (r-1)(r-3) = 0$

roots: $r = 1, 3$. General soln:

$$y(x) = c_1 e^{x} + c_2 e^{3x}.$$

Solve for c_1, c_2: $y' = c_1 e^{x} + 3c_2 e^{3x}$

$y(0) = c_1 + c_2 = 7 \Rightarrow 2c_2 = 4 \Rightarrow c_2 = 2$

$y'(0) = c_1 + 3c_2 = 11$

$\cancel{c_1 + 2 = 11 \Rightarrow c_1 = \frac{5}{2}}$

$\cancel{c_1 =}$

$\cancel{y(x) = \frac{5}{2} e^{x} + \frac{1}{2} e^{3x}}$

$c_1 + 6 = 11 \Rightarrow c_1 = 5$

$$y(x) = 5e^{x} + 2e^{3x}$$