Solutions for H.W. 1

Sec. 1.1 (10) $x^2y'' + xy' - y = \ln x$, $y_1 = x - \ln x$, $y_2 = \frac{1}{x} - \ln x$.

$y_1' = 1 - \frac{1}{x}$, $y_1'' = (1-\frac{1}{x})' = x^{-2}$ or $\frac{1}{x^2}$

$\Rightarrow x^2 y_1'' + x y_1' - y_1 = x^2 \cdot \frac{1}{x^2} + x(1-\frac{1}{x}) - (x - \ln x)$

$= 1 + x - 1 - x + \ln x$

$= \ln x \Rightarrow y_1$ satifies equation.

$y_2' = (\frac{1}{x} - \ln x)' = -\frac{1}{x^2} - \frac{1}{x}$

$y_2'' = (-x^{-2} - x^{-1})' = 2x^{-3} + x^{-2}$

$\Rightarrow x^2 y_2'' + x y_2' - y_2 = x^2(2x^{-3} + x^{-2}) + x(-\frac{1}{x^2} - \frac{1}{x}) - (\frac{1}{x} - \ln x)$

$= 2x^{-1} + 1 - \frac{1}{x} - 1 - \frac{1}{x} + \ln x$

$= \ln x \Rightarrow y_2$ solves the equation.

(27) The slope of the graph of $y = g(x)$ at (x,y) is the sum of x and y.

The slope of the graph of $y = g(x)$ at x, y is just $\frac{dy}{dx}$ (i.e. $g'(x)$). Therefore we must have

$\boxed{\frac{dy}{dx} = x + y}$ at the point (x, y)

(43) a) $\frac{dx}{dt} = kx^2$. Show $x(t) = \frac{1}{C-kt}$ is

a general solution where C is an arbitrary constant.

$$\frac{dx}{dt} = \left(\frac{1}{C-kt}\right)' = \frac{d}{dt}(C-kt)^{-1} = -(C-kt)^{-2} \cdot (-k)$$

$$= \frac{k}{(C-kt)^2}$$

But $kx(t)^2 = k\left[\frac{1}{C-kt}\right]^2 = \frac{k}{(C-kt)^2} = \frac{dx}{dt}$

so $\frac{dx}{dt} = kx^2$.

b) $\frac{dx}{dt} = kx^2$, $x(0) = 0$.

By "inspection" the function $x(t) = 0$ for all t is a solution. Certainly $\frac{dx}{dt} = 0$ all t and $kx(t) = 0$ all t so $\frac{dx}{dt} = kx^2$. Also $x(0) = 0$.

Sec. 1.2 (16) $a(t) = \frac{1}{\sqrt{t+4}}$, $v_0 = -1$, $x_0 = 1$

$v(t) = \int a(t)\,dt + C = \int \frac{1}{\sqrt{t+4}}\,dt + C$. Sub. $u = t+4$, $du = dt$

$$\Rightarrow v = \int \frac{du}{u^{1/2}} + C = \int u^{-1/2}\,du + C = 2u^{1/2} + C = 2\sqrt{t+4} + C$$

$v(0) = 2\sqrt{4} + C = 4 + C = -1 \Rightarrow C = -5$

$$\Rightarrow \quad v(t) = 2\sqrt{t+4} - 5$$

$$x(t) = \int v(t)\,dt + c = \int 2\sqrt{t+4} - 5\,dt + c$$

$$= 2\int \sqrt{t+4}\,dt - 5t + c$$

$$= 2\cdot\frac{2}{3}(t+4)^{3/2} - 5t + c$$

$$x(0) = \frac{4}{3}4^{3/2} + C = 1 \Rightarrow \frac{32}{3} + C = 1, \quad C = 1 - \frac{32}{3} = -\frac{29}{3}$$

$$\Rightarrow \boxed{x(t) = \frac{4}{3}(t+4)^{3/2} - 5t - \frac{29}{3}}$$

26 $\quad v_0 = 100\ m/s, \quad x_0 = 20m \quad$ (take time $t=0$ when projectile is launched) We know that $x = -\frac{g}{2}t^2 + v_0 t + x_0$

$$\Rightarrow \quad x = -\frac{9.8}{2}t^2 + 100t + 20 \qquad (g = 9.8\ m/s^2)$$

a) It's max height occurs when $v(t) = 0 = x'(t)$

$$\Rightarrow \quad x'(t) = -9.8t + 100 = 0 \Rightarrow t = \frac{100}{9.8}$$

So max height will be $x\left(\frac{100}{9.8}\right) = -\frac{9.8}{2}\left(\frac{100}{9.8}\right)^2 + 100\cdot\left(\frac{100}{9.8}\right) + 20$

$$= \frac{10{,}000}{19.6} + 2 = \boxed{530.2m}$$

b) It will pass the top of the building

when ~~to~~ $X(t) = 20 \Rightarrow$

$$-\frac{9.8}{2}t^2 + 100t + \cancel{20} = \cancel{20}$$

$$-\frac{9.8}{2}t^2 + 100t = 0 \Rightarrow t\left[100 - \frac{9.8}{2}t\right] = 0$$

$$\Rightarrow \cancel{t}\ 100 - \frac{9.8}{2}t = 0$$ ($t = 0$ when it is launched)

$$\Rightarrow \frac{9.8}{2}t = 100 \Rightarrow \boxed{t = \frac{200}{9.8}\ \text{s.}}$$ or $\boxed{20.41\ \text{s.}}$

c) total time in air: When object strikes ground or

$X(t) = 0$:

$$-\frac{9.8}{2}t^2 + 100t + 20 = 0.$$

$$-9.8t^2 + 200t + 40 = 0$$

$$9.8t^2 - 200t - 40 = 0$$

Use Quad. Formula

take positive root.

$$\boxed{t = \frac{200 + \sqrt{200^2 - 4(9.8)(-40)}}{2(9.8)}}$$

$\boxed{\text{or } 20.61\ \text{s.}}$

(33) $x'' = -g$. $x = -\frac{g}{2}t^2 + v_0 t + x_0$ in general.

Need to find g for "planet Gzyx".

~~t=20~~ If $x_0 = 20$, $v_0 = 0$ then when $t=2$

$x(t) = 0$ (given). Therefore

$$-\frac{g}{2}(2)^2 + 0 \cdot 2 + 20 = 0 = x(2)$$

$$-2g + 20 = 0 \Rightarrow \boxed{g = 10 \text{ f/s}^2}$$

Now, if dropped from 200 f. building $x_0 = 200$,

$v_0 = 0$ so

$$x(t) = -\frac{10}{2}t^2 + 200 = -5t^2 + 200.$$

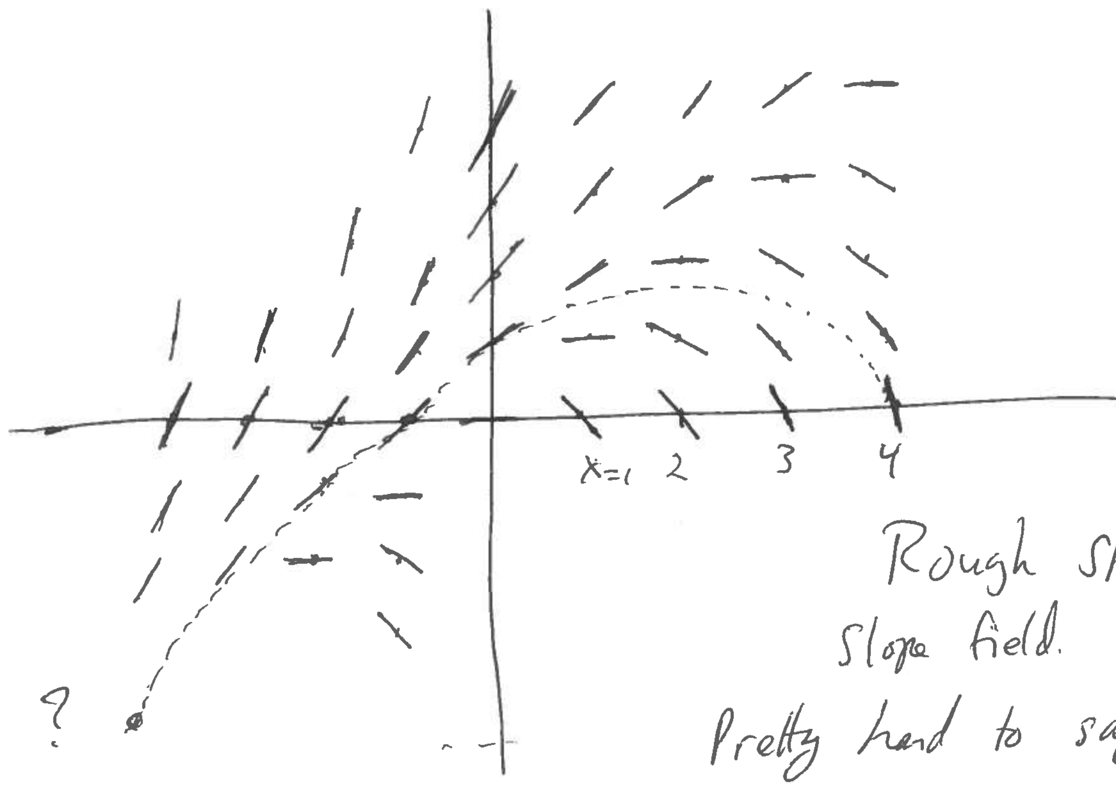
Will hit ground when $x=0 \Leftrightarrow -5t^2 + 200 = 0$

$$t^2 = \frac{200}{5} = 40 \Rightarrow t = \sqrt{40} = \boxed{2\sqrt{10} \text{ s.}}$$ (take $t>0$; after dropped)

$x' = -10t$ so speed when hits is

$$|x'(2\sqrt{10})| = 10 \cdot 2\sqrt{10} = \boxed{20\sqrt{2} \text{ f/s}}$$

Sec 1.3 (22) $y' = y - x$, $y(4) = 0$ $y(-4) = ?$



Rough Sketch of Slope field.
Pretty hard to say but
$y(-4) \approx -4$?

(27) $y(x) = \begin{cases} 0 & x \le c \\ (x-c)^2 & x > c \end{cases}$

a) If $x < c \Rightarrow y(x) = 0$ (all x nearby)

$\Rightarrow y'(x) = 0$

and $2\sqrt{y(x)} = 2\sqrt{0} = 0 \Rightarrow y' = 2\sqrt{y}$

If $x > c$ then $y'(x) = 2(x-c)$

$2\sqrt{y} = 2\sqrt{(x-c)^2} = 2(x-c) = y'$

$\Rightarrow y' = 2\sqrt{y}$

At $x=c$: $\lim\limits_{h\to 0^+} \frac{y(c+h)-y(c)}{h} = \lim\limits_{h\to 0^+} \frac{(c+h-c)^2-0}{h}$

$= \lim\limits_{h\to 0^+} \frac{h^2}{h} = \lim\limits_{h\to 0^+} h = 0$

$\lim\limits_{h\to 0^-} \frac{y(c+h)-y(c)}{h} = \lim\limits_{h\to 0^-} \frac{0-0}{h} = 0$

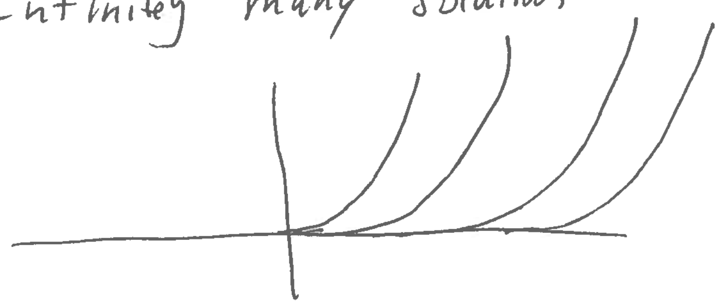
$\Rightarrow$ y differentiable at c and $y'(c)=0$

But $2\sqrt{y(c)} = 2\sqrt{0} = 0$. Thus $y'(x) = 2\sqrt{y(x)}$

all x.

<u>Note</u>: For any $c \geq 0$, $y(x)$ is a solution of the initial value problem $\begin{array}{l} y' = 2\sqrt{y} \\ y(0)=0 \end{array}$

Infinitely many solutions



all solutions.

b) If <u>$b<0$</u> then $y' = 2\sqrt{y}$, $y(0)=b$ will have no solutions since $2\sqrt{b}$ will not be a real number.

If $y(0)=0$, infinitely many soln.'s (see above)

(29)

$$y(x) = \begin{cases} 0 & x \leq c \\ (x-c)^3 & x > c \end{cases} \qquad y' = 3y^{2/3}$$

For $x < c$: $\quad y'(x) = 0' = 0$ all x

$$2y^{2/3}(x) = 2 \cdot 0 = 0 \Rightarrow y' = 3y^{2/3}$$

For $x > c$: $\quad y'(x) = 3(x-c)^2$

$$2y^{2/3} = 2\left[(x-c)^3\right]^{2/3} = 2(x-c)^2 = y'$$

For $x = c$: $$\lim_{h\to 0^+} \frac{y(c+h) - y(c)}{h} = \lim_{h\to 0^+} \frac{(c+h-c)^3 - 0}{h} = \lim_{h\to 0^+} \frac{h^3}{h} = 0$$

$$\lim_{h\to 0^-} \frac{y(c+h) - y(c)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$$

$\Rightarrow$ $y'(c)$ exists and $y'(c) = 0$. But

$2(y(c))^{2/3} = 2 \cdot 0 = 0$ so $y' = 3y^{2/3}$ all x.

Note: If pieced together with $y = x^3$ as drawn in the text:

$$y(x) = \begin{cases} x^3 & x \leq 0 \\ 0 & 0 < x \leq c \\ (x-c)^3 & c < x \end{cases}$$

you will get a soln: For $x < 0$

$y'(x) = 3x^2$. $\quad 3y^{2/3} = 3(x^3)^{2/3} = 3x^2 = y'(x)$.

At $x = 0$, you can check as above that $y'(0) = 0. = 2y(0)^{2/3}$

Note $\begin{array}{l} y' = 3y^{2/3} \\ y(a) = b \end{array}$ will always have at least one solution for any (a,b), by the discussion preceding problem 11 since $f(x,y) = 3y^{2/3}$ is continuous.

From the above example $\begin{array}{l} y' = 3y^{2/3} \\ y(0) = 0 \end{array}$ has infinitely many solutions. This does not contradict Theorem 1 since $\frac{\partial f}{\partial y} = 2y^{-1/3}$ is not continuous when $y = 0$.

If $y(a) = b$ where $b \neq 0$, there will be a unique solution on <u>some</u> interval containing a since $3y^{2/3}$, $2y^{-1/3}$ are continuous at (a,b).

Sec. 1.4 | 6 $\dfrac{dy}{dx} = 3\sqrt{xy} \Rightarrow \dfrac{dy}{dx} = 3\sqrt{x}\sqrt{y}$

$$\frac{dy}{\sqrt{y}} = 3\sqrt{x}\,dx \Rightarrow \int \frac{dy}{y^{1/2}} = 3\int x^{1/2}dx + C$$

$$\int y^{-1/2}dy = 3\cdot\frac{2}{3}x^{3/2} + C \Rightarrow 2y^{1/2} = 2x^{3/2} + C$$

$$\Rightarrow y^{1/2} = x^{3/2} + C \Rightarrow \boxed{y = (x^{3/2} + C)^2}$$

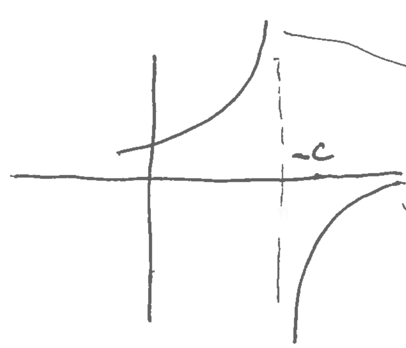
29 a) $\dfrac{dy}{dx} = y^2 \Rightarrow \dfrac{1}{y^2}dy = dx$

$$\int \frac{1}{y^2}dy = \int dx + C$$

$$-y^{-1} = x + C \Rightarrow \boxed{y = -\frac{1}{x+C}}$$

b) Note that $y(x) = 0$ all x is a soln. since $y'(x) = 0$ all x and $y^2(x) = 0$ all x. But $y(x) = 0$ all x not equal to $-\dfrac{1}{x+C}$ for any C.

c) Actually for all points (a,b) ~~...~~ $y' = y^2$, $y(a) = b$ will have a unique solution by Theorem 1.



Only one of these is a solution on an interval.

(34) $\frac{dP}{dt} = kP$. From class we have solution $P(t) = P_0 e^{kt}$ for some constants $P_0 = P(0)$ and k. In 10 hours (after some initial time which we'll call $t=0$) $P(t)$ increased by a factor of 6. Thus

$$P(10) = 6P(0) \Rightarrow \cancel{P_0} e^{k\cdot 10} = 6\cancel{P_0}$$

$$\Rightarrow e^{k10} = 6 \Rightarrow k10 = \ln 6 \quad \text{or} \quad \boxed{k = \frac{1}{10}\ln 6}$$

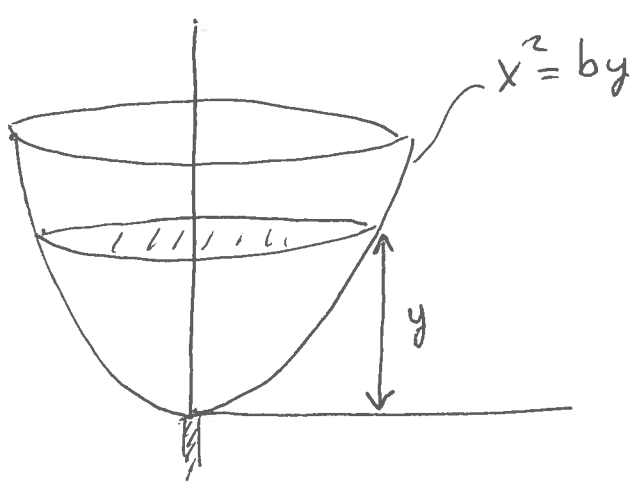
To double: $P(T) = 2P_0$ (T = time to double)

$$\Rightarrow \cancel{P_0} e^{kT} = 2\cancel{P_0} \quad \text{or}$$

$$e^{\frac{1}{10}\ln 6 \cdot T} = 2 \Rightarrow \frac{1}{10}\ln 6 \cdot T = \ln 2$$

Solve for T: $\boxed{T = 10\frac{\ln 2}{\ln 6}}$

(59)



y = water depth

$t=0$: noon

$y(0) = 4$ f.

t = time in seconds

$y(3600) = 1$ f.

$g = 32 f/s^2$

The equation from Torricelli's Law is

$$A(y)\,\frac{dy}{dt} = -a\sqrt{2gy} \qquad a = \text{area of hole}$$

at y, $A(y) = \cancel{4}\pi x^2 = 2\pi b y$ (since $x^2 = by$)

So

$$\cancel{4}\pi b y\,\frac{dy}{dt} = -a\sqrt{64}\; y^{1/2} = -8a\,y^{1/2}$$

$$y^{1/2}\,dy = -\frac{8a}{\pi b}\,dt\,, \quad \int y^{1/2}\,dy = -\int \frac{4a}{\pi b}\,dt$$

$$\frac{2}{3}\,y^{3/2} = -\frac{8a}{\pi b}\,t + C$$

$$y^{3/2} = -\frac{12a}{\pi b}\,t + C$$

$$y = \left(-\frac{12a}{\pi b}\,t + C\right)^{2/3} \qquad y(0) = 4 = C^{2/3},\quad C = 4^{3/2} = 8$$

$$y = \left(8 - \frac{12a}{\pi b}\,t\right)^{2/3}, \quad y(3600) = 1$$

$$\Rightarrow\ 1 = \left(8 - \frac{12a}{\pi b}\cdot 3600\right)^{2/3} \Rightarrow 8 - \frac{12}{\pi}\cdot\frac{a}{b}\cdot 3600 = 1$$

$$\frac{6}{\pi}\cdot\frac{a}{b}\cdot 3600 = 7 \Rightarrow \frac{12a}{\pi b} = \frac{7}{3600} \quad \left(\text{Solving for } \frac{6a}{\pi b}\right)$$

$$\Rightarrow\ \boxed{y = \left(8 - \frac{7t}{3600}\right)^{2/3}} \quad \text{or, in hours } \boxed{y = (8-7t)^{2/3}}$$

ⓐ

ⓑ Tank will be empty when $y = 0 \iff 8 - \dfrac{7t}{3600} = 0$

$$\iff t = \frac{8\cdot 3600}{7} \text{ or after } \frac{8}{7}\text{ hr.} \approx \boxed{1.08 \\ -1.09\text{ pm}}$$

c. Initially, $y=4$, $x=2$ $\quad x^2 = by \Leftrightarrow 4 = b4$

$\Leftrightarrow \boxed{b=1}$

From above $\frac{12a}{\pi b} = \frac{7}{3600} \Rightarrow a = \frac{7 \cdot \pi}{12 \cdot 3600} =$

$\Rightarrow \pi r^2 = \frac{7\pi}{3600 \cdot 6} \Rightarrow r^2 = \frac{7}{6 \cdot 3600} \Rightarrow r = \sqrt{\frac{7}{12 \cdot 3600}}$

$\Rightarrow \boxed{r = \frac{1}{60}\sqrt{\frac{7}{12}} \text{ ft.}}$

Sec. 1.5 (11) $\quad xy' + y = 3xy, \quad y(1) = 0$

$\Rightarrow y' + \frac{1}{x} y = 3y \Rightarrow y' + (\frac{1}{x} - 3)y = 0$

$P(x) = \frac{1}{x} - 3 \qquad \rho(x) = e^{\int P dx} = e^{\int \frac{1}{x} - 3\, dx} = e^{\ln|x| - 3x}$

$\rho(x) = e^{\ln|x|} e^{-3x} = |x| e^{-3x}$. $\quad x > 0$ since initial

condition is when $x = 1$:

$\rho(x) = x e^{-3x}$. $\quad$ Multiply equation by $\rho(x)$

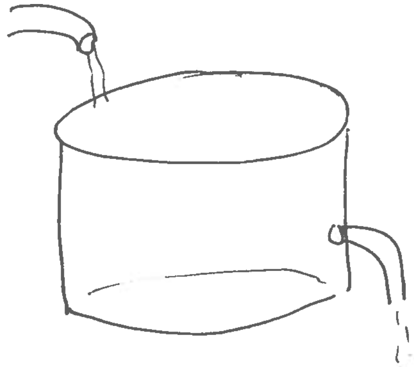
$(x e^{-3x} y)' = 0 \quad$ Integrate both sides

$x e^{-3x} y = C \quad C$ a constant.

$y = \frac{C e^{3x}}{x}$. $\quad y(1) = C e^3 = 0 \Rightarrow C = 0$

$\boxed{y = 0 \text{ all } x}$

(33)



$r_i = 5\,L/s$ incoming vol. rate

$c_i = 0$ (incoming concentration.

r_o = outgoing vol. rate $= 5\,L/s$

c_o = outgoing concentration $= \dfrac{x}{V}$

Here $V(t) =$ constant since $r_i = r_o$, $V = 1000$

$$\frac{dx}{dt} = \cancel{r_i c_i} - r_o c_o = -5\frac{x}{V} = \frac{-5}{1000}x = -\frac{x}{200}$$

$$\frac{dx}{dt} = -\frac{1}{200}x \Rightarrow x = Ce^{-\frac{1}{200}t}$$

$x(0) = 100.$ $\qquad x(0) = C = 100$

$$\boxed{x(t) = 100e^{-\frac{1}{200}t}}$$

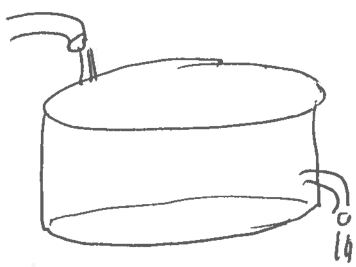
Solve for $x(t) = 10 \Rightarrow 100e^{-\frac{1}{200}t} = 10$

$e^{\frac{1}{200}t} = 10 \Rightarrow \frac{1}{200}t = \ln 10$ $\quad \boxed{t = 200\ln 10}$

or 460s

or ≈ 7 min. 41 s.

(37)



$r_i = 5$ gal/s $\qquad r_o = 3$ gal/s $\qquad c_o = \dfrac{x}{V(t)}$

$c_i = 1$ lb./gal. $\qquad V(0) = 100$ gal

V increasing at const rate of 2 gal./s $\Rightarrow V(t) = 100 + 2t$

$x(0) = 50$ lb.

$$\frac{dx}{dt} = r_i c_i - r_o c_o = 5\cdot 1 - 3\cdot\frac{x}{V(t)}$$

$$\frac{dx}{dt} + \frac{3}{100+2t} x = 5$$

$x(0) = 50$ $\quad P(t) = \dfrac{3}{100+2t}$, $\int P\,dt = 3\int \dfrac{1}{100+2t}\,dt =$

$= \dfrac{3}{2} \ln(100+2t)$

$$\rho(t) = e^{\int P\,dt} = e^{3/2 \ln(100+2t)} = e^{\ln(100+2t)^{3/2}} = (100+2t)^{3/2}$$

$$\left((100+2t)^{3/2} x\right)' = 5(100+2t)^{3/2}$$

$$(100+2t)^{3/2} x = 5\int (100+2t)^{3/2} dt + C \qquad \begin{array}{l} u = 100+2t \\ du = 2dt \end{array}$$

$$= \frac{5}{2}\int u^{3/2} du + C = u^{5/2} + C$$

$$= (100+2t)^{5/2} + C$$

$$x = (100+2t) + C(100+2t)^{-3/2}$$

$$x(0) = 100 + C(100)^{-3/2} = 50$$

$$100 + \frac{C}{1000} = 50 \Rightarrow C = -50{,}000$$

$$x = (100+2t) - 50{,}000(100+2t)^{-3/2}$$

The tank is full when $V(t) = 100 + 2t = 400$ or $t = 150$

$$x = 400 - 50{,}000(400)^{-3/2} = 400 - 6.25 = 393.75 \text{ lb}$$