

Semester Project Part 2: Due April 19, 2019

Instructions. Please prepare your own report on 8x11 paper, handwritten. Work alone or in groups. Help is available by telephone, office visit or email. All problems in Part 2 of the semester project reference Chapters 3, 7 in Edwards-Penney. Use the sample problems with solutions to fully understand the required details:

<http://www.math.utah.edu/~gustafso/s2019/2280/quiz/sampleQuizzes/project-part2.pdf>

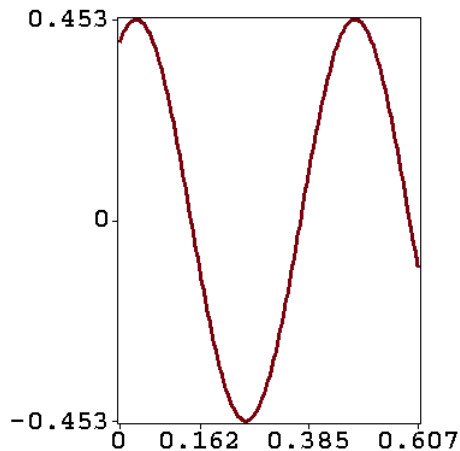
Visit the Math Center in building LCB for assistance on problem statements, references and technical details.

Problem 1. Harmonic Vibration

A mass of $m = 200$ grams attached to a spring of Hooke's constant k undergoes free undamped vibration. At equilibrium, the spring is stretched 10 cm by a force of 4 Newtons. At time $t = 0$, the spring is stretched 0.4 m and the mass is set in motion with initial velocity 3 m/s directed away from equilibrium. Find:

- (a) The numerical value of Hooke's constant k .
- (b) The initial value problem for vibration $x(t)$.
- (c) Show details for solving the initial value problem for $x(t)$.

The answer is $x(t) = \frac{2}{5} \cos(\sqrt{200}t) + \frac{3}{20}\sqrt{2} \sin(\sqrt{200}t)$, graphed below.



Problem 2. Harmonic Vibration, continued.

Assume results (a), (b), (c) from Problem 1 above. In particular, assume

$$x(t) = \frac{2}{5} \cos(\sqrt{200}t) + \frac{3\sqrt{2}}{20} \sin(\sqrt{200}t).$$

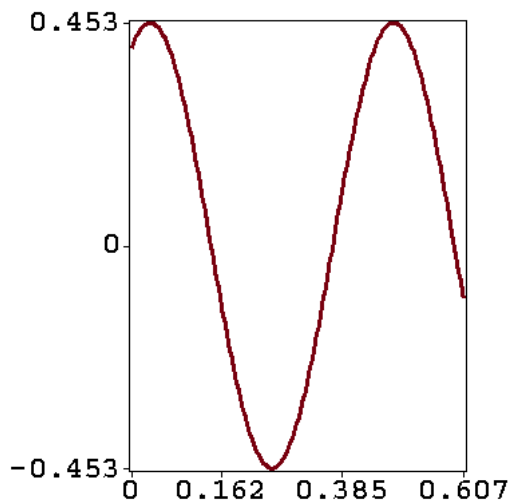
Complete these parts.

(d) Plot the solution $x(t)$ using technology, approximately matching the graphic below.

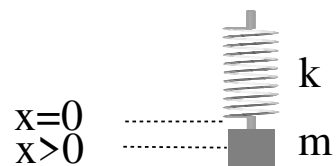
(e) Show trig details for conversion of $x(t)$ to phase-amplitude form

$$x(t) = \frac{\sqrt{82}}{20} \cos(\sqrt{200}t - \arctan(3\sqrt{2}/8)).$$

(f) Report from the answer in part (e) decimal values for the period, amplitude and phase angle. Two-place decimal accuracy is sufficient.



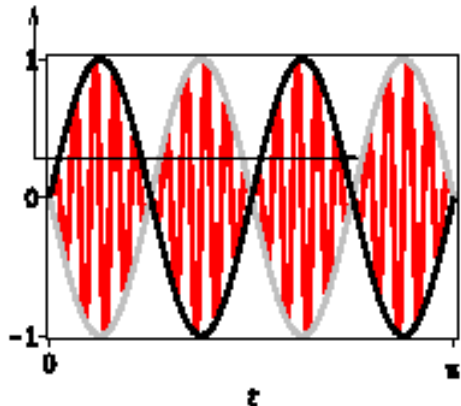
Problem 3. Undamped Spring-Mass System



A mass of 6 Kg is attached to a spring that elongates 40 centimeters due to a force of 12 Newtons. The motion starts at equilibrium with velocity -10 m/s. Find an equation for $x(t)$ using the free undamped vibration model $mx'' + kx = 0$.

Problem 4. Beats

The physical phenomenon of **beats** refers to the periodic interference of two sound waves of slightly different frequencies. A destructive interference occurs during a very brief interval, so our impression is that the sound periodically stops, only briefly, and then starts again with a beat, a section of sound that is instantaneously loud again. An illustration of the graphical meaning appears in the figure below.



Beats

Shown in red is a periodic oscillation $x(t) = 2 \sin 4t \sin 40t$ with rapidly-varying factor $\sin 40t$ and the two slowly-varying envelope curves $x_1(t) = 2 \sin 4t$ (black), $x_2(t) = -2 \sin 4t$ (grey).

The undamped, forced spring-mass problem $x'' + 1296x = 640 \cos(44t)$, $x(0) = x'(0) = 0$ has by trig identities the solution $x(t) = \cos(36t) - \cos(44t) = 2 \sin 4t \sin 40t$.

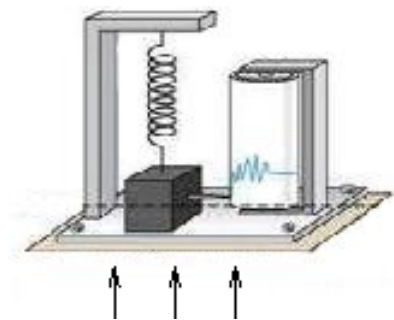
The Problem. Solve the initial value problem

$$x'' + 1444x = 1056 \cos(50t), \quad x(0) = x'(0) = 0$$

by undetermined coefficients and linear algebra, obtaining the solution $x(t) = \cos(38t) - \cos(50t)$. Then show the trig details for $x(t) = 2 \sin(6t) \sin(44t)$. Finally, graph $x(t)$ and its slowly varying envelope curves on $0 \leq t \leq \pi$.

Problem 5. Vertical Motion Seismoscope

The 1875 **horizontal motion seismoscope** of F. Cecchi (1822-1887) reacted to an earthquake. It started a clock, and then it started motion of a recording surface, which ran at a speed of 1 cm per second for 20 seconds. The clock provided the observer with the earthquake hit time.



A Simplistic Vertical Motion Seismoscope

The apparently stationary heavy mass on a spring writes with the attached stylus onto a rotating drum, as the ground moves up. The generated trace is $x(t)$.

The motion of the heavy mass m in the figure can be modeled initially by a forced spring-mass system with damping. The initial model has the form

$$mx'' + cx' + kx = f(t)$$

where $f(t)$ is the vertical ground force due to the earthquake. In terms of the vertical ground motion $u(t)$, we write via Newton's second law the force equation $f(t) = -mu''(t)$ (compare to falling body $-mg$). The final model for the motion of the mass is then

$$(1) \quad \begin{cases} x''(t) + 2\beta\Omega_0 x'(t) + \Omega_0^2 x(t) = -u''(t), \\ \frac{c}{m} = 2\beta\Omega_0, \quad \frac{k}{m} = \Omega_0^2, \\ x(t) = \text{center of mass position measured from equilibrium,} \\ u(t) = \text{vertical ground motion due to the earthquake.} \end{cases}$$

Terms **seismoscope**, **seismograph**, **seismometer** refer to the device in the figure. Some observations:

Slow ground movement means $x' \approx 0$ and $x'' \approx 0$, then (1) implies $\Omega_0^2 x(t) = -u''(t)$. The seismometer records ground acceleration.

Fast ground movement means $x \approx 0$ and $x' \approx 0$, then (1) implies $x''(t) = -u''(t)$. The seismometer records ground displacement.

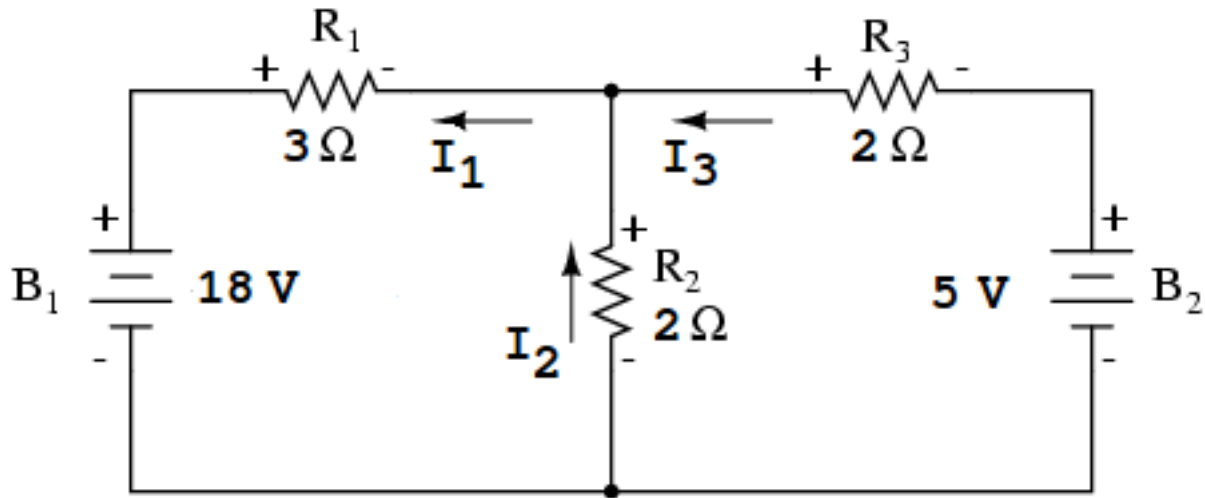
A **release test** begins by starting a vibration with u identically zero. Two successive maxima $(t_1, x_1), (t_2, x_2)$ are recorded. This experiment determines constants β, Ω_0 .

The objective of (1) is to determine $u(t)$, by knowing $x(t)$ from the seismograph.

The Problem.

Assume the seismograph trace can be modeled at time $t = 0$ (a time after the earthquake struck) by $x(t) = 10 \cos(3t)$. Assume a release test determined $2\beta\Omega_0 = 16$ and $\Omega_0^2 = 80$. Explain how to find a formula for the ground motion $u(t)$, then provide details for the answer $u(t) = \frac{710}{9} \cos(3t) - \frac{160}{3} \sin(3t)$ (assume integration constants are zero).

Problem 6. Resistive Network with 2 Loops and DC Sources.



The **Branch Current Method** can be used to find a 3×3 linear system for the **branch currents** I_1, I_2, I_3 .

$$\begin{array}{rclcl} I_1 - I_2 - I_3 & = & 0 & \text{KCL, upper node} \\ 3I_1 + 2I_2 & = & 18 & \text{KVL, left loop} \\ 2I_2 - 2I_3 & = & 5 & \text{KVL, right loop} \end{array}$$

Symbol **KCL** means *Kirchhoff's Current Law*, which says the algebraic sum of the currents at a node is zero. Symbol **KVL** means *Kirchhoff's Voltage Law*, which says the algebraic sum of the voltage drops around a closed loop is zero.

(a) Solve the equations to find the currents I_1, I_2, I_3 in the figure.

(b) Compute the voltage drops across resistors R_1, R_2, R_3 . Answer: $\frac{93}{8}, \frac{51}{8}, \frac{11}{8}$ volts.

(c) Replace the 5 volt battery by a 4 volt battery. Solve the system again, and report the new currents and voltage drops.

References. Edwards-Penney 3.7, electric circuits. All About Circuits Volume I – DC, by T. Kuphaldt:

<http://www.allaboutcircuits.com/>.

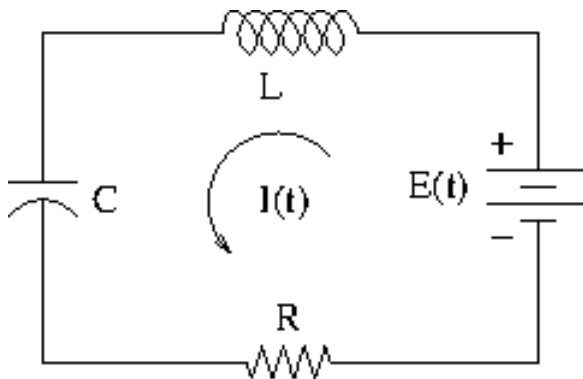
Course slides on Electric Circuits:

<http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf>

Solved examples of electrical networks can be found in the lecture notes of Ruye Wang:

<http://fourier.eng.hmc.edu/e84/lectures/ch2/node2.html>.

Problem 7. RLC-Circuit



The Problem. Suppose $E = \sin(40t)$, $L = 1$ H, $R = 50\ \Omega$ and $C = 0.01$ F. The model for the charge $Q(t)$ is $LQ'' + RQ' + \frac{1}{C}Q = E(t)$.

- (a) Differentiate the charge model and substitute $I = \frac{dQ}{dt}$ to obtain the current model $I'' + 50I' + 100I = 40 \cos(40t)$.
- (b) Find the **reactance** $S = \omega L - \frac{1}{\omega C}$, where $\omega = 40$ is the input frequency, the natural frequency of $E = \sin(40t)$ and $E' = 40 \cos(40t)$. Then find the **impedance** $Z = \sqrt{S^2 + R^2}$.
- (c) The steady-state current is $I(t) = A \cos(40t) + B \sin(40t)$ for some constants A, B . Substitute $I = A \cos(40t) + B \sin(40t)$ into the current model (a) and solve for A, B .
Answers: $A = -\frac{6}{625}$, $B = \frac{8}{625}$.
- (d) Write the answer in (c) in phase-amplitude form $I = I_0 \sin(40t - \delta)$ with $I_0 > 0$ and $\delta \geq 0$. Then compute the **time lag** δ/ω .
Answers: $I_0 = 0.016$, $\delta = \arctan(0.75)$, $\delta/\omega = 0.0160875$.
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References

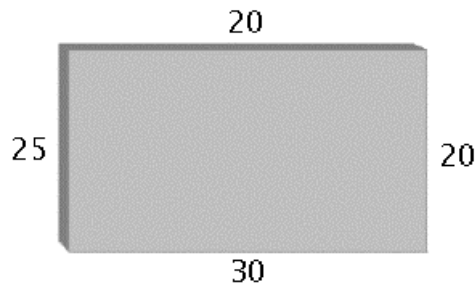
Course slides on Electric Circuits:

<http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/electricalCircuits.pdf>

Edwards-Penney *Differential Equations and Boundary Value Problems*, sections 3.4, 3.5, 3.6, 3.7.

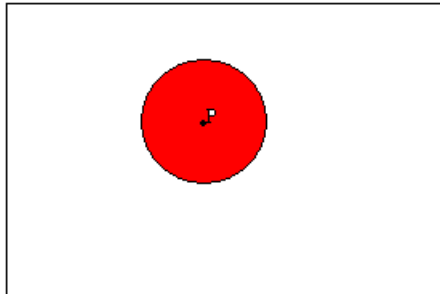
Problem 8. Heat Transfer and the Mean Value Property.

Consider the cross section of a long rectangular dam on a river, represented in the figure.



The boundaries of the dam are subject to three factors: the temperature in degrees Celsius of the air (20), the water (25), and the ground at its base (30).

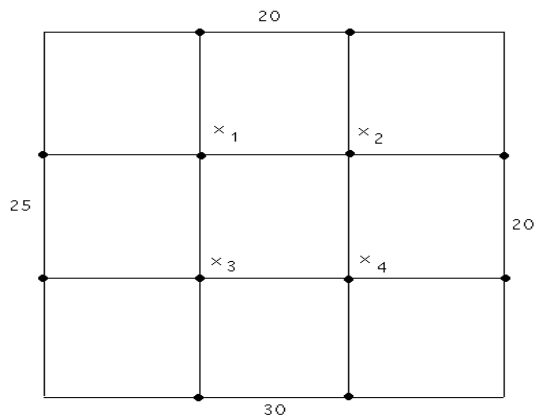
An analysis of the heat transfer from the three sources will be done from the equilibrium temperature, which is found by the Mean Value Property below.



The Mean Value Property

If a plate is at thermal equilibrium, and C is a circle with center P contained in the plate, then the temperature at P is the average value of the temperature function over the boundary of C .

A version of the Mean Value Property says that the temperature at center P of circle C is the average of the temperatures at four equally-spaced points on C . We construct a grid as in the figure below, label the unknown temperatures at interior grid points as x_1, x_2, x_3, x_4 , then use the property to obtain four equations.



Four-Point Temperature Averages

$$x_1 = \frac{1}{4}(20 + 25 + x_2 + x_3)$$

$$x_2 = \frac{1}{4}(20 + 20 + x_1 + x_4)$$

$$x_3 = \frac{1}{4}(25 + 30 + x_1 + x_4)$$

$$x_4 = \frac{1}{4}(20 + 30 + x_2 + x_3)$$

The Problem

(a) Solve the equations for the four temperatures $x_1 = 23.125, x_2 = 21.875, x_3 = 25.625, x_4 = 24.375$. Use technology.

(b) Replace the temperatures 20, 25, 30 by 20, 16, 12 and re-compute x_1 to x_4 .

(c) Using the temperatures from part (b), subdivide the grid to make it 6×6 . Assign 25 unknown temperatures x_i at the interior grid points. Find equations for x_1 to x_{25} and solve using technology.

References. EPH Chapters 12, 13, used for Partial Differential Equations 3150. The corresponding material in the 2280 book by Edwards–Penney in in Chapter 9. A basic reference about steady–state heat applications is <http://cecs.wright.edu/~stthomas/htchapter03.pdf>

Chris Tisdell has an elementary YouTube video on the subject, in which he discusses the Mean Value Property: https://www.youtube.com/watch?v=p60dU_62KcQ

Problem 8 Alternate. Archeology and the Dot Product.

Complete this problem in place of Mean Value Property Problem 8, if you prefer archeology to heat conduction. Credit applies to only one Problem 8. There is no Sample Problem for this archeology example.

Archeologist Sir Flinders Petrie collected and analyzed pottery fragments from 900 Egyptian graves. He deduced from the data an historical ordering of the 900 sites. Petrie's ideas will be illustrated for 4 sites and 3 pottery types. The matrix rows below represent sites 1, 2, 3, 4 and the matrix columns represent pottery types 1, 2, 3. A matrix entry is 1 if the site has that pottery type and 0 if not. This is an **incidence matrix**.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Petrie Matrix. It is an incidence matrix in which the ones in each column appear together, like the matrix above.

Counting Pottery Types. The dot product of row 2 and row 3 is

$$(0, 1, 1) \cdot (1, 0, 1) = 0 * 1 + 1 * 0 + 1 * 1 = 1$$

which means sites 2 and 3 have one pottery type in common. **Please pause on this arithmetic**, until you agree that the products $0 * 1, 1 * 0, 1 * 1$ add to the number of pottery types in common.

Sites with pottery in common are expected to be historically close in time. Because pottery types evolve, old types cease production when newly created pottery types begin production, which gives meaning to the clustered ones in the columns of A .

The Problem. Find a sequence of row swaps which starts with the incidence matrix

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and ends with a Petrie matrix A . Express the swaps as elementary matrices E_1, E_2, \dots and write A as the product of elementary matrices times C . There is not a unique set of swaps, because of duplicate rows.

You may use Kendall's ideas with Robinson matrices or the simple definition of a Petrie matrix, given above. In any case, after finding Petrie matrix A , compute Robinson's matrix $R = AA^T$ and apply the ideas in sample quiz 7, to double-check the answer.

References. Edwards-Penney *Differential Equations and Linear Algebra*, Ch 3.

David Kendall's 1969 work on incidence matrices, interval graphs and seriation in archeology: <https://projecteuclid.org/euclid.pjm/1102983306>

W. S. Robinson, a method for chronologically ordering archeological deposits, in the April 1951 issue of **American Antiquity**: <http://www.jstor.org/stable/276978>

Incidence matrix C was invented as an archeology exercise on **Seriation** of 7 Maryland sites based on 3 types of historical ceramics:

<http://www.saa.org/publicftp/public/primarydocuments/Seriation.pdf>

Problem 9. Solving Higher Order Constant-Coefficient Equations

The **Algorithm** applies to constant-coefficient homogeneous linear differential equations of order N , for example equations like

$$y'' + 16y = 0, \quad y'''' + 4y'' = 0, \quad \frac{d^5 y}{dx^5} + 2y''' + y'' = 0.$$

1. Find the N th degree characteristic equation by Euler's substitution $y = e^{rx}$. For instance, $y'' + 16y = 0$ has characteristic equation $r^2 + 16 = 0$, a polynomial equation of degree $N = 2$.
2. Find all real roots and all complex conjugate pairs of roots satisfying the characteristic equation. List the N roots according to multiplicity.
3. Construct N distinct Euler solution atoms from the list of roots. Then the general solution of the differential equation is a linear combination of the Euler solution atoms with arbitrary coefficients c_1, c_2, c_3, \dots .

The solution space is then $S = \text{span}(\text{the } N \text{ Euler solution atoms})$.

Examples: Constructing Euler Solution Atoms from roots.

Three roots $0, 0, 0$ produce three atoms $e^{0x}, xe^{0x}, x^2e^{0x}$ or $1, x, x^2$.

Three roots $0, 0, 2$ produce three atoms e^{0x}, xe^{0x}, e^{2x} .

Two complex conjugate roots $2 \pm 3i$ produce two atoms $e^{2x} \cos(3x), e^{2x} \sin(3x)$.

Explained. The Euler substitution $y = e^{rx}$ produces a solution of the differential equation when r is a complex root of the characteristic equation. Complex exponentials are not used directly. Ever. They are replaced by sines and cosines times real exponentials, which are Euler solution atoms. Euler's formula $e^{i\theta} = \cos \theta + i \sin \theta$ implies $e^{2x} \cos(3x) = \frac{e^{2x} e^{3xi} + e^{2x} e^{-3xi}}{2} = \frac{1}{2} e^{2x+3xi} + \frac{1}{2} e^{2x-3xi}$, which is a linear combination of complex exponentials, solutions of the differential equation because of Euler's substitution. Superposition implies $e^{2x} \cos(3x)$ is a solution. Similar for $e^{2x} \sin(3x)$. The independent pair $e^{2x} \cos(3x), e^{2x} \sin(3x)$ replaces both $e^{(2+3i)x}$ and $e^{(2-3i)x}$.

Four complex conjugate roots listed according to multiplicity as $2 \pm 3i, 2 \pm 3i$ produce four atoms $e^{2x} \cos(3x), e^{2x} \sin(3x), xe^{2x} \cos(3x), xe^{2x} \sin(3x)$.

Seven roots $1, 1, 3, 3, 3, \pm 3i$ produce seven atoms $e^x, xe^x, e^{3x}, xe^{3x}, x^2e^{3x}, \cos(3x), \sin(3x)$.

Two conjugate complex roots $a \pm bi$ ($b > 0$) arising from roots of $(r-a)^2 + b^2 = 0$ produce two atoms $e^{ax} \cos(bx), e^{ax} \sin(bx)$.

The Problem

Solve for the general solution or the particular solution satisfying initial conditions.

(a) $y'' + 4y' = 0$

(b) $y'' + 4y = 0$

(c) $y''' + 4y' = 0$

(d) $y'' + 4y = 0, y(0) = 1, y'(0) = 2$

(e) $y'''' + 81y'' = 0, y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$

(f) The characteristic equation is $(r+1)^2(r^2-1) = 0$.

(g) The characteristic equation is $(r-1)^2(r^2-1)^2((r+1)^2+9) = 0$.

(h) The characteristic equation roots, listed according to multiplicity, are $0, 0, -1, 2, 2, 3+4i, 3-4i, 3+4i, 3-4i$.

Problem 10. Laplace Theory

Laplace theory collects theorems and transform tables to implement the *method of quadrature* for higher order differential equations, linear systems of differential equations, and certain partial differential equations.

Laplace's method *solves differential equations*.

Laplace's Quadrature Method: multiply the equation by the Laplace integrator $e^{-st} dt$ and then integrate across the equation $t = 0$ to $t = \infty$.

Laplace's Method: multiply across the equation by symbol \mathcal{L} , then manipulate the result as though \mathcal{L} is a matrix. Solve for the unknown(s) using Laplace tables and Laplace theorems.

Laplace Theory uses properties and tables, and almost never the **Direct Laplace Transform** $F(s) = \int_0^\infty f(t)e^{-st} dt$. See Sample Problem 10 for techniques and ideas which apply to the problems below.

The Problem. Solve by table methods or Laplace's method.

(a) Forward table. Find $\mathcal{L}(f(t))$ for $f(t) = 3(t+1)^2e^{2t} + 2e^t \sin(3t)$.

(b) Backward table. Find $f(t)$ for

$$\mathcal{L}(f(t)) = \frac{4s}{s^2 + 4} + \frac{s - 1}{s^2 - 2s + 5}.$$

(c) Solve the initial value problem $x''(t) + 2x'(t) + 5x(t) = e^t$, $x(0) = 0$, $x'(0) = 1$.