Semester Project Part 1: Due March 1, 2019

Instructions. Please prepare your own report on 8x11 paper, handwritten. Work alone or in groups. Help is available by telephone, office visit or email. All problems in Part 1 of the semester project reference Chapters 1, 2 in Edwards-Penney. Use the sample problems with solutions to fully understand the required details:
`http://www.math.utah.edu/~gustafso/s2019/2280/quiz/sampleQuizzes/project-part1.pdf`
Visit the Math Center in building LCB for assistance on problem statements, references and technical details.

Problem 1. An answer check for the differential equation and initial condition

$$\frac{dy}{dx} = k(73 - y(x)), \quad y(0) = 28 \tag{1}$$

requires substitution of the candidate solution $y(x) = 73 - 45\, e^{-kx}$ into the left side (LHS) and right side (RHS), then compare the expressions for equality for all symbols. The process of testing LHS = RHS applies to both the differential equation and the initial condition, making the answer check have **two** presentation panels. Complete the following:

1. Show the two panels in an answer check for initial value problem (1).

2. Relate (1) to a Newton cooling model for warming a 28 F frozen ice cream bar to room temperature 73 F.

3. Let x be the time in minutes. Find the Newton cooling constant k, given the additional information that the ice cream bar reaches 33 F in 5 minutes.

References. Edwards-Penney sections 1.1, 1.4, 1.5. Newton cooling in Serway and Vuille, *College Physics 9/E*, Brooks-Cole (2011), ISBN-10: 0840062060. Newton cooling differential equation $\frac{du}{dt} = -h(u(t) - u_1)$, Math 2280 slide **Three Examples**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250ThreeExamples.pdf`
Math 2280 slide on **Answer checks**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/FTC-Method-of-Quadrature.pdf`

Problem 2. A 2-ft high conical water urn drains from an orifice 6 inches above the base. The tank drains according to the Torricelli model

$$|y(x)|^2 \frac{dy}{dx} = -0.021\sqrt{|y(x)|}, \quad y(0) = y_0. \tag{2}$$

Symbol $y(x) \geq 0$ is the tank water height in feet above the orifice at time x seconds, while $y_0 \geq 0$ is the water height at time $x = 0$.

Establish these facts about the physical problem.

1. If $y_0 > 0$, then the solution $y(x)$ is uniquely determined and computable by numerical software. Justify using Picard's existence-uniqueness theorem.

2. Solve equation (2) using separation of variables when y_0 is 19 inches, then numerically find the drain time. Check your answer with technology.

References. Edwards-Penney, Picard's theorem 1 section 1.3 and Torricelli's Law section 1.4. Tank draining `Mathematica` demo at **Wolfram Research**:
http://demonstrations.wolfram.com/TimeToDrainATankUsingTorricellisLaw/
Carl Schaschke, *Fluid Mechanics: Worked Examples for Engineers*, The Institution of Chemical Engineers (2005), ISBN-10: 0852954980, Chapter 6.
Math 2280 slide on **Picard and Peano Theorems**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/Picard+DirectionFields.pdf`
Manuscript on applications of first order equations Example 35:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250SciEngApplications.pdf,`

Problem 3. Three applications for the Newton cooling equation $y' = -h(y - y_1)$ are considered, where h, y_1 are constants.

(a) **Cooling**. An apple initially 23 degrees Celsius is placed in a refrigerator at 2 degrees Celsius. The exponential model is the apple temperature $u(t) = 2 + 21e^{-ht}$. Display the differential equation and the initial condition.

(b) **Heating**. A beef roast initially 8 degrees Celsius is placed in an oven at 190 degrees Celsius. The exponential model is the roast temperature $u(t) = 190 - 182e^{-ht}$. Display the differential equation and the initial condition.

(c) **Fish Length**. K. L. von Bertalanffy in 1934 modeled the growth of fish using the equation $\frac{dL}{dt} = h(L_\infty - L(t))$. The fish has mature length L_∞ inches, length $L(t)$ while growing, t is in months and h is the growth rate. Given growth data of $L(0) = 0$, $L(1) = 5$, $L(2) = 7$, find the mature length L_∞, the growth rate h and the months to grow to 95% of mature length.

References. Edwards-Penney section 1.5.
Course notes on **Newton's linear drag model**:
http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250kinetics.pdf
Course notes on **Newton cooling**:
http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/linearapplications2-5.pdf
Wikipedia biography of **Ludwig von Bertalanffy**:
http://en.wikipedia.org/wiki/Ludwig_von_Bertalanffy
Pisces Conservation Ltd **Growth Models**, especially Gompertz, logistic and von Bertalanffy.
http://www.pisces-conservation.com/growthhelp/index.html
Serway and Vuille, *College Physics 9/E*, Brooks-Cole (2011), ISBN-10: 0840062060.
The **Coffee Cooling Problem**, a Wolfram Demonstration by S.M. Binder.
http://demonstrations.wolfram.com/TheCoffeeCoolingProblem/

Problem 4. Logistic growth $F(x) = rx(1 - x/M)$ can be used to describe the annual natural growth of a fish stock. Symbol $x(t)$ is the stock biomass in number of fish at the start of month t. The intrinsic growth rate is symbol r. The environmental carrying capacity in stock biomass terms is symbol M.

(a) Assume a pond has carrying capacity $M = 780$ thousand fish. If 92% of the the fish survive to maturity, then $r = 0.92$. Display the no-harvesting model $x'(t) = F(x(t))$, using only symbols x and t.

(b) Assume constant harvesting $H \geq 0$ to give the model $x'(t) = F(x(t)) - H$. Use the college algebra quadratic formula to find the equilibrium points in terms of symbols r, M, H. Then verify facts **A, B, C** from your answer.

b-1. If $H = \frac{rM}{4}$, then there is one equilibrium point $x = \frac{M}{2}$ (a double real root).

b-2. If $H > \frac{rM}{4}$, then there is no equilibrium point.

b-3. If $0 < H < \frac{rM}{4}$, then there are two equilibrium points.

(c) Replace symbols r, M by 0.92 and 780. Create a short filmstrip of 5 hand-drawn phase diagrams for the equation $x'(t) = F(x(t)) - H$ using the successive harvest values

$$H = 0, \frac{1}{4}\left(\frac{rM}{4}\right), \frac{1}{2}\left(\frac{rM}{4}\right), \frac{1}{1}\left(\frac{rM}{4}\right), \frac{11}{10}\left(\frac{rM}{4}\right).$$

Each phase diagram shows the equilibria and at least 5 threaded solutions, with labels for funnel, spout and node. The graph window is $t = 0$ to 36 months and $x = 0$ to $2M$.

(d) Justify a guess for the **maximum sustainable harvest**, based on your 5 diagrams. This is an approximate value for the largest catch H that can be taken over 36 months.

References. Edwards-Penney sections 2.1, 2.2.

Course document on the **Logistic Equation**:

http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250logistic.pdf

Course document on **Stability**:

http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250phaseline.pdf

Course document on **Fish Farming**:

. http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/fishFarming2014.pdf.

A logistic fish farming investigation in Malaysia by **M.F. Laham 2012**:

http://www.ukm.my/jsm/pdf_files/SM-PDF-41-2-2012/04%20Mohamed%20Faris.pdf

Problem 5. A graphic called a **phase diagram** displays the behavior of all solutions of $u' = F(u)$. A **phase line diagram** is an abbreviation for a direction field on the vertical axis (u-axis). It consists of equilibrium points and signs of $F(u)$ between equilibria. A phase diagram can be created solely from a phase line diagram, using just three drawing rules:

1. Solutions don't cross.

2. Equilibrium solutions are horizontal lines $u = c$. All other solutions are increasing or decreasing.

3. A solution curve can be moved rigidly left or right to create another solution curve.

Use these tools on the equation $u' = (u-1)(u-2)^2(u+2)$ to make a phase line diagram, and then make a phase diagram with at least 8 threaded solutions. Label the equilibria as stable, unstable, funnel, spout, node.

References. Edwards-Penney section 2.2.
Course document on **Stability**,:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/2250phaseline.pdf`

Problem 6. An autonomous differential equation $\frac{dy}{dx} = F(x)$ with initial condition $y(0) = y_0$ has a formal solution

$$y(x) = y_0 + \int_0^x F(u)du.$$

The integral may not be solvable by calculus methods. In this case, the integral is evaluated numerically to compute $y(x)$ or to plot a graphic. There are three basic numerical methods that apply, the rectangular rule (RECT), the trapezoidal rule (TRAP)and Simpson's rule (SIMP).

Apply the three methods for $F(x) = \cos(x^2)$ and $y_0 = 0$ using step size $h = 0.2$ from $x = 0$ to $x = 1$. Then fill in the blanks in the following table. Use technology if it saves time. Lastly, compare the four data sets in a plot, using technology.

$x-$ values	0.0	0.2	0.4	0.6	0.8	1.0
$y-$ to 10 digits	0.0	0.1999680024	0.3989772129	0.5922705167	0.7678475376	0.9045242379
$y-$ RECT values	0.0	0.2	0.3998400213	0.5972854780		0.9448839943
$y-$ TRAP values	0.0	0.1999200107	0.3985627497		0.7646744186	0.8989142250
$y-$ SIMP values	0.0	0.1999666703	0.3989746144	0.5922670741	0.7678445414	

References. Edwards-Penney Sections 2.4, 2.5, 2.6, because methods Euler, Modified Euler and RK4 reduce to RECT, TRAP, SIMP methods when $f(x, y)$ is independent of y, i.e., an equation $y' = F(x)$.
Course document on numerical solution of $y' = F(x)$, **RECT, TRAP, SIMP methods**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/solve-quadrature-numerically.pdf`
Wolfram Alpha at `http://www.wolframalpha.com/` can do the RECT rule and graphics with input string

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integrate cos(x^2) using left endpoint method with interval width 0.2 from
x=0 to x=1
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Problem 7. Picard–Lindelöf Theorem and Spring-Mass Models

Picard-Lindelöf Theorem. Let $\vec{f}(x, \vec{y})$ be defined for $|x - x_0| \le h$, $\|\vec{y} - \vec{y}_0\| \le k$, with $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ continuous. Then for some constant H, $0 < H < h$, the problem

$$\begin{cases} \vec{y}\,'(x) &= \vec{f}(x, \vec{y}(x)), \quad |x - x_0| < H, \\ \vec{y}(x_0) &= \vec{y}_0 \end{cases}$$

has a unique solution $\vec{y}(x)$ defined on the smaller interval $|x - x_0| < H$.

Emile Picard

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The Problem. The second order problem

$$\begin{cases} u'' + 2u' + 17u = 100, \\ u(0) = 1, \\ u'(0) = -1 \end{cases} \tag{1}$$

is a spring-mass model with damping and constant external force. The variables are time x in seconds and elongation $u(x)$ in meters, measured from equilibrium. Coefficients in the equation represent mass $m = 1$ kg, a viscous damping constant $c = 2$, Hooke's constant $k = 17$ and external force $F(x) = 100$.

Convert the scalar initial value problem into a vector problem, to which Picard's vector theorem applies, by supplying details for the parts below.

(a) The conversion uses the **position-velocity substitution** $y_1 = u(x), y_2 = u'(x)$, where y_1, y_2 are the invented components of vector $\vec{y}$. Then the initial data $u(0) = 1, u'(0) = -1$ converts to the vector initial data

$$\vec{y}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(b) Differentiate the equations $y_1 = u(x), y_2 = u'(x)$ in order to find the scalar system of two differential equations, known as a **dynamical system**:

$$y_1' = y_2, \quad y_2' = -17y_1 - 2y_2 + 100.$$

(c) The derivative of vector function $\vec{y}(x)$ is written $\vec{y}\,'(x)$ or $\frac{d\vec{y}}{dx}(x)$. It is obtained by componentwise differentiation: $\vec{y}\,'(x) = \begin{pmatrix} y_1' \\ y_2' \end{pmatrix}$. The vector differential equation model of scalar system (1) is

$$\begin{cases} \vec{y}\,'(x) &= \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \\ \vec{y}(0) &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}. \end{cases} \tag{2}$$

(d) System (2) fits the hypothesis of Picard's theorem, using symbols

$$\vec{f}(x, \vec{y}) = \begin{pmatrix} 0 & 1 \\ -17 & -2 \end{pmatrix} \vec{y}(x) + \begin{pmatrix} 0 \\ 100 \end{pmatrix}, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The components of vector function $\vec{f}$ are continuously differentiable in variables x, y_1, y_2, therefore $\vec{f}$ and $\frac{\partial \vec{f}}{\partial \vec{y}}$ are continuous.

References. Chapter 2, Edwards-Penney.
Course slides on the Picard and Direction Fields:
http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/Picard+DirectionFields.pdf
Course slides on the Picard Theorem:
http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/picardHigherOrderSuperposition.pdf
Course slides on the Vector Picard Theorem:
http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/picardVectorTheorem.pdf

Problem 8. The velocity of a crossbow bolt launched upward from the ground was determined from a video and a speed gun to complete the following table.

Time t in seconds	Velocity $v(t)$ in ft/sec	Location
0.000	60	Ground
1.7	0	Maximum
3.5	-52	Near Ground Impact

(a) The bolt velocity can be approximated by a quadratic polynomial

$$z(t) = at^2 + bt + c$$

which reproduces the table data. Find three equations for the coefficients a, b, c. Then solve for the coefficients.

(b) Assume a linear drag model $v' = -32 - \rho v$. Substitute the polynomial answer $v = z(t)$ of (a) into this differential equation, then substitute $t = 0$ and solve for $\rho \approx 0.11$.

(c) Solve the model $w' = -32 - \rho w$, $w(0) = 60$ with $\rho = 0.11$.

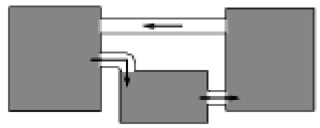
(d) The error between $z(t)$ and $w(t)$ can be measured. Is the drag coefficient value $\rho = 0.11$ reasonable?

References. Edwards-Penney sections 2.3, 3.1, 3.2.
Course documents on **Linear algebraic equations**:
http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/linearequDRAFT.pdf
Course documents on **Newton kinematics**:
http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/newtonModelsDE2008.pdf

Problem 9. Consider the system of differential equations

$$\begin{aligned} x_1' &= -\tfrac{1}{5}x_1 &&+ \tfrac{1}{7}x_3, \\ x_2' &= \tfrac{1}{5}x_1 - \tfrac{1}{3}x_2, && \\ x_3' &= \tfrac{1}{3}x_2 &&- \tfrac{1}{7}x_3, \end{aligned}$$

for the amounts x_1, x_2, x_3 of salt in recirculating brine tanks, as in the figure:



Recirculating Brine Tanks A, B, C

The volumes are $50, 30, 70$ for A, B, C, respectively.

The steady-state salt amounts in the three tanks are found by formally setting $x_1' = x_2' = x_3' = 0$ and then solving for the symbols x_1, x_2, x_3.

(a) Solve the corresponding linear system of algebraic equations for answers x_1, x_2, x_3.

(b) The total amount of salt is uniformly distributed in the tanks in ratio $5 : 3 : 7$. Explain this mathematically from the answer in (a).

References. Edwards-Penney sections 3.1, 3.2, 7.3 Figure 5.
Course documents on **Linear algebraic equations**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/linearequDRAFT.pdf`
Course documents on **Systems and Brine Tanks**:
`http://www.math.utah.edu/~gustafso/s2019/2280/lectureslides/systemsBrineTank.pdf`