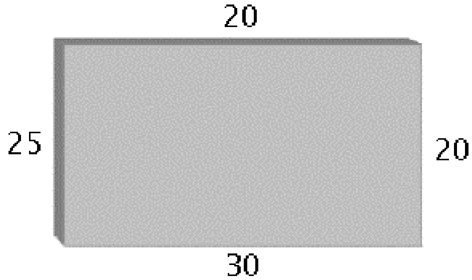
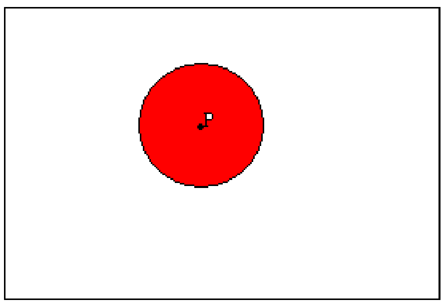
Problem 8. **Heat Transfer** and the Mean Value Property.

Consider the cross section of a long rectangular dam on a river, represented in the figure.



The boundaries of the dam are subject to three factors: the temperature in degrees Celsius of the air (20), the water (25), and the ground at its base (30).

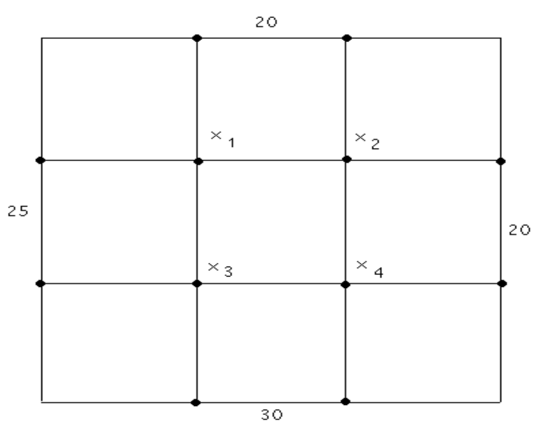
An analysis of the heat transfer from the three sources will be done from the equilibrium temperature, which is found by the Mean Value Property below.



The Mean Value Property

If a plate is at thermal equilibrium, and C is a circle with center P contained in the plate, then the temperature at P is the average value of the temperature function over the boundary of C.

A version of the Mean Value Property says that the temperature at center P of circle C is the average of the temperatures at four equally-spaced points on C. We construct a grid as in the figure below, label the unknown temperatures at interior grid points as x_1, x_2, x_3, x_4, then use the property to obtain four equations.



Four-Point Temperature Averages

$$x_1 = \frac{1}{4}(20 + 25 + x_2 + x_3)$$

$$x_2 = \frac{1}{4}(20 + 20 + x_1 + x_4)$$

$$x_3 = \frac{1}{4}(25 + 30 + x_1 + x_4)$$

$$x_4 = \frac{1}{4}(20 + 30 + x_2 + x_3)$$

The Problem

(a) Solve the equations for the four temperatures $x_1 = 23.125, x_2 = 21.875, x_3 = 25.625, x_4 = 24.375$. Use technology.

(b) Replace the temperatures $20, 25, 30$ by $20, 16, 12$ and re-compute x_1 to x_4.

(c) Using the temperatures from part (b), subdivide the grid to make it 6×6. Assign 25 unknown temperatures x_i at the interior grid points. Find equations for x_1 to x_{25} and solve using technology.

References. EPH Chapters 12, 13, used for Partial Differential Equations 3150. The corresponding material in the 2280 book by Edwards–Penney in in Chapter 9. A basic reference about steady–state heat applications is `http://cecs.wright.edu/~sthomas/htchapter03.pdf`
Chris Tisdell has an elementary YouTube video on the subject, in which he discusses the Mean Value Property: `https://www.youtube.com/watch?v=p60dU_62KcQ`

Problem 8 Alternate. Archeology and the Dot Product.

Complete this problem in place of Mean Value Property Problem 8, if you prefer archeology to heat conduction. Credit applies to only one Problem 8. There is no Sample Problem for this archeology example.

> Archeologist Sir Flinders Petrie collected and analyzed pottery fragments from 900 Egyptian graves. He deduced from the data an historical ordering of the 900 sites. Petrie's ideas will be illustrated for 4 sites and 3 pottery types. The matrix rows below represent sites 1, 2, 3, 4 and the matrix columns represent pottery types 1, 2, 3. A matrix entry is 1 if the site has that pottery type and 0 if not. This is an **incidence matrix**.
>
> $$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$
>
> **Petrie Matrix**. It is an incidence matrix in which the ones in each column appear together, like the matrix above.
>
> **Counting Pottery Types**. The dot product of row 2 and row 3 is
>
> $$\begin{pmatrix} 0, 1, 1 \end{pmatrix} \cdot \begin{pmatrix} 1, 0, 1 \end{pmatrix} = 0 * 1 + 1 * 0 + 1 * 1 = 1$$
>
> which means sites 2 and 3 have one pottery type in common. **Please pause on this arithmetic**, until you agree that the products $0 * 1, 1 * 0, 1 * 1$ add to the number of pottery types in common.
>
> Sites with pottery in common are expected to be historically close in time. Because pottery types evolve, old types cease production when newly created pottery types begin production, which gives meaning to the clustered ones in the columns of A.

The Problem. Find a sequence of row swaps which starts with the incidence matrix

$$C = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

and ends with a Petrie matrix A. Express the swaps as elementary matrices $E_1, E_2, \ldots$ and write A as the product of elementary matrices times C. There is not a unique set of swaps, because of duplicate rows.

You may use Kendall's ideas with Robinson matrices or the simple definition of a Petrie matrix, given above. In any case, after finding Petrie matrix A, compute Robinson's matrix $R = AA^T$ and apply the ideas in sample quiz 7, to double-check the answer.

References. Edwards-Penney *Differential Equations and Linear Algebra*, Ch 3.

David Kendall's 1969 work on incidence matrices, interval graphs and seriation in archeology: `https://projecteuclid.org/euclid.pjm/1102983306`

W. S. Robinson, a method for chronologically ordering archeological deposits, in the April 1951 issue of **American Antiquity**: `http://www.jstor.org/stable/276978`

Incidence matrix C was invented as an archeology exercise on **Seriation** of 7 Maryland sites based on 3 types of historical ceramics:

`http://www.saa.org/publicftp/public/primarydocuments/Seriation.pdf`