Linear Dynamical Systems

Matrix Exponential: Putzer Formula for e^{At}

Variation of Parameters and Undetermined Coefficients

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The 2×2 Matrix Exponential e^{At}

Definition. The **matrix exponential** e^{At} is the $n \times n$ matrix $\Phi(t)$ defined by

$$(1) \quad \frac{d}{dt}\Phi = A\Phi, \qquad (2) \quad \Phi(0) = I.$$

Alternatively, Φ is the augmented matrix of solution vectors for the n problems $\frac{d}{dt}\vec{v}_k = A\vec{v}_k$, $\vec{v}_k(0) =$ column k of I, $1 \le k \le n$.

Example. A 2×2 matrix A has exponential matrix e^{At} with columns equal to the solutions of the two problems

$$\begin{cases} \frac{d}{dt}\vec{\mathbf{v}}_1(t) = A\vec{\mathbf{v}}_1(t), \\ \vec{\mathbf{v}}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases} \qquad \begin{cases} \frac{d}{dt}\vec{\mathbf{v}}_2(t) = A\vec{\mathbf{v}}_2(t), \\ \vec{\mathbf{v}}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

Briefly, the 2×2 matrix $\Phi(t) = e^{At}$ satisfies the two conditions

$$(1) \quad \frac{d}{dt}\Phi(t) = A\Phi(t), \quad (2) \quad \Phi(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Putzer Matrix Exponential Formula for 2×2 Matrices

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$$

A is 2×2, $\lambda_1 \neq \lambda_2$ real.

$$e^{At} = e^{\lambda_1 t} I + t e^{\lambda_1 t}(A - \lambda_1 I)$$

A is 2×2, $\lambda_1 = \lambda_2$ real.

$$e^{At} = e^{at} \cos bt \, I + \frac{e^{at} \sin bt}{b}(A - aI)$$

A is 2×2, $\lambda_1 = \overline{\lambda}_2 = a + ib$, $b > 0$.

How to Remember Putzer's 2×2 Formula

The expressions

$$
\begin{aligned}
e^{At} &= r_1(t)I + r_2(t)(A - \lambda_1 I), \\
r_1(t) &= e^{\lambda_1 t}, \quad r_2(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}
\end{aligned} \tag{1}
$$

are enough to generate all three formulas. Fraction r_2 is the $d/d\lambda$-Newton difference quotient for r_1. Then r_2 limits as $\lambda_2 \to \lambda_1$ to the $d/d\lambda$-derivative $te^{\lambda_1 t}$. Therefore, the formula includes the case $\lambda_1 = \lambda_2$ by limiting. If $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$, then the fraction r_2 is already real, because it has for $z = e^{\lambda_1 t}$ and $w = \lambda_1$ the form

$$
r_2(t) = \frac{z - \overline{z}}{w - \overline{w}} = \frac{\sin bt}{b}.
$$

Taking real parts of expression (1) gives the complex case formula.

Variation of Parameters

Theorem 1 (Variation of Parameters for Linear Systems)
Let A be a constant $n \times n$ matrix and $\vec{\mathbf{F}}(t)$ a continuous function near $t = t_0$. The unique solution $\vec{\mathbf{x}}(t)$ of the matrix initial value problem

$$\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0,$$

is given by the **variation of parameters formula**

$$\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0 + e^{At}\int_{t_0}^{t} e^{-rA}\vec{\mathbf{F}}(r)dr. \tag{2}$$

Undetermined Coefficients

Theorem 2 (Polynomial Solutions)
Let $f(t)$ be a polynomial of degree k. Assume A is an $n \times n$ constant invertible matrix. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + f(t)\vec{\mathbf{c}}$ has a polynomial solution $\vec{\mathbf{u}}(t) = \sum_{j=0}^{k} \vec{\mathbf{c}}_j \frac{t^j}{j!}$ of degree k with vector coefficients $\{\vec{\mathbf{c}}_j\}$ given by the relations

$$\vec{\mathbf{c}}_j = -\sum_{i=j}^{k} f^{(i)}(0) A^{j-i-1}\vec{\mathbf{c}}, \quad 0 \le j \le k.$$

Changes from nth Order Undetermined Coefficients. The nth order theory using Rule I and Rule II is replaced by

> **Systems Rule for Undetermined Coefficients**. Assume $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}} + \vec{F}(t)$. Extract all Euler atoms from $\vec{F}, \vec{F}', \ldots$. Don't replace atoms by groups (Rule II). Instead, extend each existing group (Rule I) by adding $m-1$ higher power terms x^k(base atom) to the group, where m is the multiplicity of the root for the base atom in the characteristic equation $|A - rI| = 0$. The trial solution is a linear combination of the final atom list with vector coefficients.