Systems of Differential Equations
The Eigenanalysis Method

- **First Order 2×2 Systems $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$**
- **First Order 3×3 Systems $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$**
- **Second Order 3×3 Systems $\vec{\mathbf{x}}'' = A\vec{\mathbf{x}}$**
- **Vector-Matrix Form of the Solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$**
- **Four Methods for Solving a System $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$**

The Eigenanalysis Method for First Order 2×2 Systems

Suppose that A is 2×2 real and has eigenpairs

$$(\lambda_1, \vec{v}_1), \quad (\lambda_2, \vec{v}_2),$$

with $\vec{v}_1$, $\vec{v}_2$ independent. The eigenvalues λ_1, λ_2 can be both real. Also, they can be a complex conjugate pair $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$.

Theorem 1 (Eigenanalysis Method)

The general solution of $\vec{x}' = A\vec{x}$ is

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2.$$

Solving 2×2 Systems $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ with Complex Eigenvalues

If the eigenvalues are complex conjugates, then the real part $\vec{\mathbf{w}}_1$ and the imaginary part $\vec{\mathbf{w}}_2$ of the solution $e^{\lambda_1 t}\vec{\mathbf{v}}_1$ are independent solutions of the differential equation. Then the general solution in *real form* is given by the relation

$$\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{w}}_1(t) + c_2\vec{\mathbf{w}}_2(t).$$

The Eigenanalysis Method for First Order 3×3 Systems

Suppose that A is 3×3 real and has eigenpairs

$$(\lambda_1, \vec{\mathbf{v}}_1), \quad (\lambda_2, \vec{\mathbf{v}}_2), \quad (\lambda_3, \vec{\mathbf{v}}_3),$$

with $\vec{\mathbf{v}}_1$, $\vec{\mathbf{v}}_2$, $\vec{\mathbf{v}}_3$ independent. The eigenvalues λ_1, λ_2, λ_3 can be all real. Also, there can be one real eigenvalue λ_3 and a complex conjugate pair of eigenvalues $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$.

Theorem 2 (Eigenanalysis Method)
The general solution of $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ with 3×3 real A can be written as

$$\vec{\mathbf{x}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + c_2 e^{\lambda_2 t}\vec{\mathbf{v}}_2 + c_3 e^{\lambda_3 t}\vec{\mathbf{v}}_3.$$

Solving 3×3 Systems $\vec{x}' = A\vec{x}$ with Complex Eigenvalues

If there are complex eigenvalues $\lambda_1 = \overline{\lambda}_2$, then the real general solution is expressed in terms of independent solutions

$$\vec{w}_1 = \mathcal{R}\mathrm{e}(e^{\lambda_1 t}\vec{v}_1),\ \vec{w}_2 = \mathcal{I}\mathrm{m}(e^{\lambda_1 t}\vec{v}_1)$$

as the linear combination

$$\vec{x}(t) = c_1\vec{w}_1(t) + c_2\vec{w}_2(t) + c_3 e^{\lambda_3 t}\vec{v}_3.$$

The Eigenanalysis Method for Second Order Systems

Theorem 3 (Second Order Systems)
Let A be real and 3×3 with three negative eigenvalues $\lambda_1 = -\omega_1^2$, $\lambda_2 = -\omega_2^2$, $\lambda_3 = -\omega_3^2$. Let the eigenpairs of A be listed as

$$(\lambda_1, \vec{v}_1), \ (\lambda_2, \vec{v}_2), \ (\lambda_3, \vec{v}_3).$$

Then the general solution of the second order system $\vec{x}''(t) = A\vec{x}(t)$ is

$$\begin{aligned}\vec{x}(t) \ = \ & \left(a_1 \cos \omega_1 t + b_1 \frac{\sin \omega_1 t}{\omega_1}\right) \vec{v}_1 \\ & + \left(a_2 \cos \omega_2 t + b_2 \frac{\sin \omega_2 t}{\omega_2}\right) \vec{v}_2 \\ & + \left(a_3 \cos \omega_3 t + b_3 \frac{\sin \omega_3 t}{\omega_3}\right) \vec{v}_3\end{aligned}$$

Vector-Matrix Form of the Solution of $\vec{\mathrm{x}}' = A\vec{\mathrm{x}}$

The solution of $\vec{\mathrm{x}}' = A\vec{\mathrm{x}}$ in the 3×3 case is written in vector-matrix form

$$\vec{\mathrm{x}}(t) = \mathbf{aug}(\vec{\mathrm{v}}_1, \vec{\mathrm{v}}_2, \vec{\mathrm{v}}_3) \begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

This formula is normally used when the eigenpairs are real.

Complex Eigenvalues for a 2×2 System

When there is a complex conjugate pair of eigenvalues $\boldsymbol{\lambda_1 = \overline{\lambda}_2 = a + ib}$, $\boldsymbol{b > 0}$, then it is possible to extract a real solution $\vec{\mathbf{x}}$ from the complex formula and report a real solution. The work can be organized more efficiently using the matrix product

$$\vec{\mathbf{x}}(t) = e^{at}\,\mathrm{aug}(\mathrm{Re}(\vec{\mathbf{v}}_1), \mathrm{Im}(\vec{\mathbf{v}}_1)) \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

Complex Eigenvalues for a 3×3 System

When there is a complex conjugate pair of eigenvalues $\boldsymbol{\lambda}_1 = \overline{\boldsymbol{\lambda}}_2 = \boldsymbol{a} + i\boldsymbol{b}$, $\boldsymbol{b} > \mathbf{0}$, then a real solution $\vec{\mathbf{x}}$ can be extracted from the complex formula to report a real solution. The work is organized using the matrix product

$$\vec{\mathbf{x}}(t) \;=\; \mathbf{aug}(\boldsymbol{R}\mathbf{e}(\vec{\mathbf{v}}_1), \boldsymbol{I}\mathbf{m}(\vec{\mathbf{v}}_1), \vec{\mathbf{v}}_3) \begin{pmatrix} e^{at}\cos bt & e^{at}\sin bt & 0 \\ -e^{at}\sin bt & e^{at}\cos bt & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Four Methods for Solving a 2×2 System $\vec{u}' = A\vec{u}$

1. **First-order method**. If A is diagonal, then use growth-decay methods. If A is triangular, then use the linear integrating factor method.
2. **Cayley-Hamilton-Ziebur method**. If A is not diagonal, and $a_{12} \neq 0$, then $u_1(t)$ is a linear combination of the atoms constructed from the roots r of $\det(A - rI) = 0$. Solution $u_2(t)$ is found from the system by solving for u_2 in terms of u_1 and u_1'.
3. **Eigenanalysis method**. Assume A has eigenpairs $(\lambda_1, \vec{v}_1)$, $(\lambda_2, \vec{v}_2)$ with $\vec{v}_1$, $\vec{v}_2$ independent. Then $\vec{u}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2$.
4. **Resolvent method**. In Laplace notation, $\vec{u}(t) = L^{-1}\left((sI - A)^{-1}\vec{u}(0)\right)$. The inverse of $C = sI - A$ is found from the formula $C^{-1} = \mathbf{adj}(C)/\det(C)$. Cramer's Rule can replace the matrix inversion method.

Four Methods for Solving an $n \times n$ System $\vec{\mathbf{u}}' = A\vec{\mathbf{u}}$

1. **First-order method**. If A is diagonal, then use growth-decay methods. If A is triangular, then use the linear integrating factor method.
2. **Cayley-Hamilton-Ziebur method**. The solution $\vec{\mathbf{u}}(t)$ is a linear combination of the atoms constructed from the roots r of $\det(A - rI) = 0$,

$$\vec{\mathbf{u}}(t) = (\text{atom}_1)\vec{\mathbf{d}}_1 + \cdots + (\text{atom}_n)\vec{\mathbf{d}}_n.$$

   To solve for the constant vectors $\vec{\mathbf{d}}_j$, differentiate the formula $n-1$ times, then use $A^k\vec{\mathbf{u}}(t) = \vec{\mathbf{u}}^{(k+1)}(t)$ and set $t = 0$, to obtain a system for $\vec{\mathbf{d}}_1, \ldots, \vec{\mathbf{d}}_n$.
3. **Eigenanalysis method**. Assume A has eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1), \ldots, (\lambda_n, \vec{\mathbf{v}}_n)$ with $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$ independent. Then $\vec{\mathbf{u}}(t) = c_1 e^{\lambda_1 t}\vec{\mathbf{v}}_1 + \cdots + c_n e^{\lambda_n t}\vec{\mathbf{v}}_n$.
4. **Resolvent method**. In Laplace notation, $\vec{\mathbf{u}}(t) = L^{-1}\left((sI - A)^{-1}\vec{\mathbf{u}}(0)\right)$. The inverse of $C = sI - A$ is found from the formula $C^{-1} = \mathbf{adj}(C)/\det(C)$. Cramer's Rule can replace the matrix inversion method.