Stability of Dynamical systems

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Stability

Consider an autonomous system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ with $\vec{\mathbf{f}}$ continuously differentiable in a region D in the plane.

Stable equilibrium. An equilibrium point $\vec{\mathbf{u}}_0$ in D is said to be **stable** provided for each $\epsilon > 0$ there corresponds $\delta > 0$ such that **(a)** and **(b)** hold:

(a) Given $\vec{\mathbf{u}}(0)$ in D with $\|\vec{\mathbf{u}}(0) - \vec{\mathbf{u}}_0\| < \delta$, then $\vec{\mathbf{u}}(t)$ exists on $0 \le t < \infty$.

(b) Inequality $\|\vec{\mathbf{u}}(t) - \vec{\mathbf{u}}_0\| < \epsilon$ holds for $0 \le t < \infty$.

Unstable equilibrium. The equilibrium point $\vec{\mathbf{u}}_0$ is called **unstable** provided it is **not stable**, which means **(a)** or **(b)** fails (or both).

Asymptotically stable equilibrium. The equilibrium point $\vec{\mathbf{u}}_0$ is said to be **asymptotically stable** provided **(a)** and **(b)** hold (it is **stable**), and additionally

(c) $\lim_{t\to\infty} \|\vec{\mathbf{u}}(t) - \vec{\mathbf{u}}_0\| = 0$ for $\|\vec{\mathbf{u}}(0) - \vec{\mathbf{u}}_0\| < \delta$.

Isolated equilibria

An autonomous system is said to have an **isolated equilibrium** at $\vec{u} = \vec{u}_0$ provided $\vec{u}_0$ is the only constant solution of the system in $|\vec{u} - \vec{u}_0| < r$, for $r > 0$ sufficiently small.

Theorem 1 (Isolated Equilibrium)
The following are equivalent for a constant planar system $\vec{u}'(t) = A\vec{u}(t)$:

1. The system has an isolated equilibrium at $\vec{u} = \vec{0}$.
2. $\det(A) \neq 0$.
3. The roots λ_1, λ_2 of $\det(A - \lambda I) = 0$ satisfy $\lambda_1 \lambda_2 \neq 0$.

Proof: The expansion $\det(A - \lambda I) = (\lambda_1 - \lambda)(\lambda_2 - \lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1 \lambda_2$ shows that $\det(A) = \lambda_1 \lambda_2$. Hence $\mathbf{2} \equiv \mathbf{3}$. We prove now $\mathbf{1} \equiv \mathbf{2}$. If $\det(A) = 0$, then $A\vec{u} = \vec{0}$ has infinitely many solutions $\vec{u}$ on a line through $\vec{0}$, therefore $\vec{u} = \vec{0}$ is not an isolated equilibrium. If $\det(A) \neq 0$, then $A\vec{u} = \vec{0}$ has exactly one solution $\vec{u} = \vec{0}$, so the system has an isolated equilibrium at $\vec{u} = \vec{0}$.

Classification of Isolated Equilibria

For linear equations

$$\vec{\mathbf{u}}'(t) = A\vec{\mathbf{u}}(t),$$

we explain the phase portrait classifications

spiral, center, saddle, node

near an isolated equilibrium point $\vec{\mathbf{u}} = \vec{0}$, and how to detect these classifications, when they occur.

Symbols λ_1, λ_2 are the roots of $\det(A - \lambda I) = 0$.

Euler solution atoms corresponding to roots λ_1, λ_2 happen to classify the phase portrait as well as its stability. A **shortcut** will be explained to determine a classification, *based only on the atoms*.

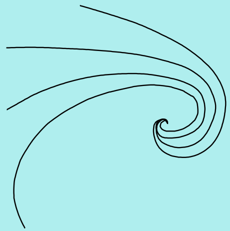

Figure 1. Spiral

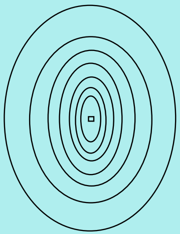

Figure 2. Center

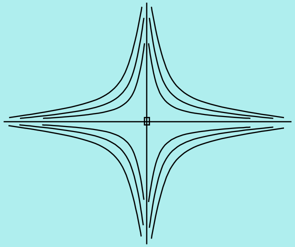

Figure 3. Saddle

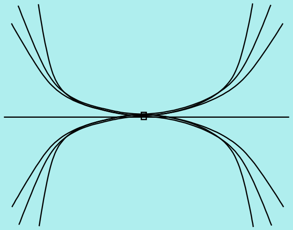

Figure 4. Improper node

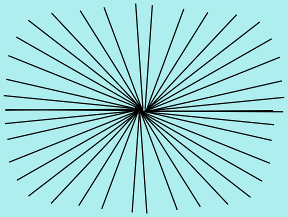

Figure 5. Proper node

Spiral $\boldsymbol{\lambda}_1 = \overline{\boldsymbol{\lambda}}_2 = \boldsymbol{a} + i\boldsymbol{b}$ complex, $\boldsymbol{a} \neq 0$, $\boldsymbol{b} > 0$.

A **spiral** has solution formula

$$\vec{\mathbf{u}}(t) = e^{at}\cos(bt)\,\vec{\mathbf{c}}_1 + e^{at}\sin(bt)\,\vec{\mathbf{c}}_2,$$

$$\vec{\mathbf{c}}_1 = \vec{\mathbf{u}}(0), \quad \vec{\mathbf{c}}_2 = \frac{A - aI}{b}\,\vec{\mathbf{u}}(0).$$

All solutions are bounded harmonic oscillations of natural frequency $\boldsymbol{b}$ times an exponential amplitude which grows if $\boldsymbol{a} > 0$ and decays if $\boldsymbol{a} < 0$. An orbit in the phase plane **spirals out** if $\boldsymbol{a} > 0$ and **spirals in** if $\boldsymbol{a} < 0$.

Center $\lambda_1 = \overline{\lambda}_2 = a + ib$ complex, $a = 0$, $b > 0$

A **center** has solution formula

$$\vec{\mathbf{u}}(t) = \cos(bt)\,\vec{\mathbf{c}}_1 + \sin(bt)\,\vec{\mathbf{c}}_2,$$

$$\vec{\mathbf{c}}_1 = \vec{\mathbf{u}}(0), \quad \vec{\mathbf{c}}_2 = \frac{1}{b}\,A\vec{\mathbf{u}}(0).$$

All solutions are bounded harmonic oscillations of natural frequency b. Orbits in the phase plane are periodic closed curves of period $2\pi/b$ which encircle the origin.

Saddle λ_1, λ_2 real, $\lambda_1\lambda_2 < 0$

A **saddle** has solution formula

$$\vec{u}(t) = e^{\lambda_1 t}\vec{c}_1 + e^{\lambda_2 t}\vec{c}_2,$$

$$\vec{c}_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\vec{u}(0), \quad \vec{c}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\vec{u}(0).$$

The phase portrait shows two lines through the origin which are tangents at $t = \pm\infty$ for all orbits.

A saddle is **unstable** at $t = \infty$ and $t = -\infty$, due to the limits of the atoms $e^{r_1 t}$, $e^{r_2 t}$ at $t = \pm\infty$.

Node λ_1, λ_2 real, $\lambda_1\lambda_2 > 0$

The solution formulas are

$$\vec{u}(t) = e^{\lambda_1 t}\left(\vec{a}_1 + t\vec{a}_2\right), \quad \text{when} \quad \lambda_1 = \lambda_2,$$

$$\vec{a}_1 = \vec{u}(0), \quad \vec{a}_2 = (A - \lambda_1 I)\vec{u}(0),$$

$$\vec{u}(t) = e^{\lambda_1 t}\vec{b}_1 + e^{\lambda_2 t}\vec{b}_2, \quad \text{when} \quad \lambda_1 \neq \lambda_2,$$

$$\vec{b}_1 = \frac{A - \lambda_2 I}{\lambda_1 - \lambda_2}\vec{u}(0), \quad \vec{b}_2 = \frac{A - \lambda_1 I}{\lambda_2 - \lambda_1}\vec{u}(0).$$

Definition 1 (node)

A **node** is defined to be an equilibrium point (x_0, y_0) such that

1. Either $\lim_{t\to\infty}(x(t), y(t)) = (x_0, y_0)$ or else $\lim_{t\to-\infty}(x(t), y(t)) = (x_0, y_0)$, for all initial conditions $(x(0), y(0)$ close to (x_0, y_0).
2. For each initial condition $(x(0), y(0))$ near (x_0, y_0), there exists a straight line L through (x_0, y_0) such that $(x(t), y(t))$ is **tangent** at $t = \infty$ to L. Precisely, L has a tangent vector $\vec{v}$ and $\lim_{t\to\infty}(x'(t), y'(t)) = c\vec{v}$ for some constant c.

Node Subclassification

Proper Node. Also called a **Star Node**.

Matrix $\boldsymbol{A}$ is required to have two eigenpairs $(\boldsymbol{\lambda}_1, \vec{\boldsymbol{v}}_1)$, $(\boldsymbol{\lambda}_2, \vec{\boldsymbol{v}}_2)$ with $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2$.

Then $\vec{\boldsymbol{u}}(0)$ in $\boldsymbol{R}^2 = \mathbf{span}(\vec{\boldsymbol{v}}_1, \vec{\boldsymbol{v}}_2)$ implies

$$\vec{\boldsymbol{u}}(0) = \boldsymbol{c}_1\vec{\boldsymbol{v}}_1 + \boldsymbol{c}_2\vec{\boldsymbol{v}}_2 \quad \text{and} \quad \vec{\boldsymbol{a}}_2 = (\boldsymbol{A} - \boldsymbol{\lambda}_1\boldsymbol{I})\vec{\boldsymbol{u}}(0) = \vec{\mathbf{0}}.$$

Therefore, $\vec{\boldsymbol{u}}(\boldsymbol{t}) = \boldsymbol{e}^{\boldsymbol{\lambda}_1\boldsymbol{t}}\vec{\boldsymbol{a}}_1$ implies trajectories are tangent to the line through $(0, 0)$ in direction $\vec{\boldsymbol{v}} = \vec{\boldsymbol{a}}_1/|\vec{\boldsymbol{a}}_1|$.

Because $\vec{\boldsymbol{u}}(0) = \vec{\boldsymbol{a}}_1$ is arbitrary, $\vec{\boldsymbol{v}}$ can be any direction, which explains the star-like phase portrait.

Node Subclassification

Improper Node with One Eigenpair

The non-diagonalizable case is also called a **Degenerate Node**.

Matrix $\boldsymbol{A}$ is required to have just one eigenpair $(\boldsymbol{\lambda}_1, \vec{\boldsymbol{v}}_1)$ and $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2$. Then $\vec{\boldsymbol{u}}'(t) = (\vec{\boldsymbol{a}}_2 + \boldsymbol{\lambda}_1\vec{\boldsymbol{a}}_1 + t\boldsymbol{\lambda}_1\vec{\boldsymbol{a}}_2)e^{\lambda_1 t}$ implies $\vec{\boldsymbol{u}}'(t)/|\vec{\boldsymbol{u}}'(t)| \approx \vec{\boldsymbol{a}}_2/|\vec{\boldsymbol{a}}_2|$ at $|t| = \infty$. Matrix $\boldsymbol{A} - \boldsymbol{\lambda}_1\boldsymbol{I}$ has rank 1, hence

$$\mathbf{Image}(\boldsymbol{A} - \boldsymbol{\lambda}_1\boldsymbol{I}) = \mathbf{span}(\vec{\boldsymbol{v}})$$

for some nonzero vector $\vec{\boldsymbol{v}}$. Then $\vec{\boldsymbol{a}}_2 = (\boldsymbol{A} - \boldsymbol{\lambda}_1\boldsymbol{I})\vec{\boldsymbol{u}}(0)$ is a multiple of $\vec{\boldsymbol{v}}$.

Trajectory $\vec{\boldsymbol{u}}(t)$ is tangent to the line through $(0, 0)$ with direction $\vec{\boldsymbol{v}}$.

Node Subclassification

Improper Node with Distinct Eigenvalues

The first possibility is when matrix $\boldsymbol{A}$ has real eigenvalues with $\boldsymbol{\lambda_2 < \lambda_1 < 0}$.
The second possibility $\boldsymbol{\lambda_2 > \lambda_1 > 0}$ is left to the reader.

Then $\vec{u}'(t) = \lambda_1 \vec{b}_1 e^{\lambda_1 t} + \lambda_2 \vec{b}_2 e^{\lambda_2 t}$ implies $\vec{u}'(t)/|\vec{u}'(t)| \approx \vec{b}_1/|\vec{b}_1|$ at $t = \infty$.
In terms of eigenpairs $(\lambda_1, \vec{v}_1), (\lambda_2, \vec{v}_2)$, we compute $\vec{b}_1 = c_1 \vec{v}_1$ and $\vec{b}_2 = c_2 \vec{v}_2$ where $\vec{u}(0) = c_1 \vec{v}_1 + c_2 \vec{v}_2$.

Trajectory $\vec{u}(t)$ is tangent to the line through $(0, 0)$ with direction $\vec{v}_1$.

Attractor and Repeller

An equilibrium point is called an **attractor** provided solutions starting nearby limit to the point as $t \to \infty$.

A **repeller** is an equilibrium point such that solutions starting nearby limit to the point as $t \to -\infty$.

Terms like **attracting node** and **repelling spiral** are defined analogously.

Almost linear systems

A nonlinear planar autonomous system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ is called **almost linear** at equilibrium point $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ if there is a 2×2 matrix A and a vector function $\vec{\mathbf{g}}$ such that

$$\vec{\mathbf{f}}(\vec{\mathbf{u}}) = A(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0) + \vec{\mathbf{g}}(\vec{\mathbf{u}}),$$
$$\lim_{\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\| \to 0} \frac{\|\vec{\mathbf{g}}(\vec{\mathbf{u}})\|}{\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|} = 0.$$

The function $\vec{\mathbf{g}}$ has the same smoothness as $\vec{\mathbf{f}}$.

We investigate the possibility that a local phase diagram at $\vec{\mathbf{u}} = \vec{\mathbf{u}}_0$ for the nonlinear system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ is graphically identical to the one for the linear system $\vec{\mathbf{y}}'(t) = A\vec{\mathbf{y}}(t)$ at $\vec{\mathbf{y}} = 0$.

Jacobian Matrix

Almost linear system results will apply to **all isolated equilibria** of $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$. This is accomplished by expanding f in a Taylor series about each equilibrium point, which implies that the ideas are applicable to different choices of A and g, depending upon which equilibrium point $\vec{\mathbf{u}}_0$ was considered.

Define the **Jacobian matrix** of $\vec{\mathbf{f}}$ at equilibrium point $\vec{\mathbf{u}}_0$ by the formula

$$J = \left\langle \partial_1 \vec{\mathbf{f}}(\vec{\mathbf{u}}_0) | \partial_2 \vec{\mathbf{f}}(\vec{\mathbf{u}}_0) \right\rangle.$$

Taylor's theorem for functions of two variables says that

$$\vec{\mathbf{f}}(\vec{\mathbf{u}}) = J(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0) + \vec{\mathbf{g}}(\vec{\mathbf{u}})$$

where $\vec{\mathbf{g}}(\vec{\mathbf{u}})/\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\| \to \mathbf{0}$ as $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\| \to 0$. Therefore, for $\vec{\mathbf{f}}$ continuously differentiable, we may always take $A = J$ to obtain from the almost linear system $\vec{\mathbf{u}}'(t) = \vec{\mathbf{f}}(\vec{\mathbf{u}}(t))$ its **linearization** $y'(t) = A\vec{\mathbf{y}}(t)$.