Solutions 2019

# Final Exam Differential Equations 2280

Tuesday, 30 April 2019, 7:30am to 10:15am

**Instructions**: No calculators, notes, tables or books. No answer check is expected. A correct answer without details counts 25%.

## Chapters 1 and 2: First Order Differential Equations

**Definitions**. An **equilibrium solution** is a constant solution, found by replacing all derivatives by zero, then solve for $y$. If $y$ found by this method is not constant, then the method fails. For $y'+py=q$, the **homogeneous equation** is $y'+py=0$. An equation $y'=f(x,y)$ is **separable** provided functions $F,G$ exist such that $f(x,y)=F(x)G(y)$.

**(a)** [20%] Apply a test to the equation $y'=x+y+y^2$, showing it fails to be separable.

A

$\frac{d}{dx}(x+y+y^2)=1 \Rightarrow \frac{f_x}{f(x,y)}=\frac{1}{x+y+y^2}$, which has a $y$ term so it is not separable

The other test also fails: $\frac{d}{dy}(x+y+y^2)=2y+1 \Rightarrow \frac{f_y}{f(x,y)}=\frac{2y+1}{y^2+y+x}$ which has an $x$ term

A

**(b)** [30%] The problem $x\dfrac{dy}{dx}=xy+3x+2y+6$ is both linear and separable. It can be solved by superposition $y=y_h+y_p$, where $y_h$ is the homogeneous solution and $y_p$ is an equilibrium solution. Show details for the answers $y_h=c\,x^2\,e^x$ and $y_p=-3$.

For $y_p$, let $\frac{dy}{dx}=0 \Rightarrow xy+3x+2y+6=0 \Rightarrow x(3+y)+2y+6=0$

$\Rightarrow y_p=-3$

For $y_h$, $x\frac{dy}{dx}=xy+2y \Rightarrow x\frac{dy}{dx}=y(x+2) \Rightarrow \frac{dy}{y}=\frac{x+2}{x}dx$

$\Rightarrow \ln|y|=x+2\ln|x|+c$

$\Rightarrow y_h=e^x e^{2\ln|x|}e^c$

$\Rightarrow y_h=Cx^2e^x$

A

(c) [20%] Solve the linear homogeneous equation $x^2 \frac{dy}{dx} + 2y = xy$.

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$$X^2 \frac{dy}{dx} + 2y = xy$$

$$\Rightarrow \frac{dy}{dx} + \frac{2y - xy}{x^2} = 0 \Rightarrow \frac{dy}{dx} + \left(\frac{2-x}{x^2}\right) y = 0$$

$$\Rightarrow W = e^{\int \frac{2-x}{x^2} dx} = e^{-\ln x} e^{-2/x} = e^{-2/x}/x$$

$$\Rightarrow (e^{-2/x}/x\, y)' = 0 \Rightarrow e^{-2/x}/x\, y = C$$

$$\Rightarrow Y = Cxe^{2/x}$$

Integrating factor

A

(d) [30%] Solve $2\frac{d}{dt}v(t) = 5\,e^{-t} + \frac{1}{2}v(t)$, $v(0) = 0$ by the linear integrating factor method. Show all steps.

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$$2\frac{d}{dt}V(t) = 5e^{-t} + \frac{1}{2}V(t)$$ Given

$$\Rightarrow \frac{d}{dt}V(t) - \frac{1}{4}V(t) = 5/2\, e^{-t}$$ Rearrange eq.

$$\Rightarrow W = e^{\int -\frac{1}{4} dt} = e^{-\frac{1}{4}t}$$ Calculate int. fact.

$$\Rightarrow (e^{-\frac{1}{4}t}V(t))' = 5/2\, e^{-5/4t}$$ Int. factor method

$$\Rightarrow e^{-\frac{1}{4}t}V(t) = \int 5/2\, e^{-5/4t} dt$$ Integrate both sides

$$\Rightarrow e^{-1/4t}V(t) = -2e^{-5/4t} + C$$ Evaluate integral

$$\Rightarrow V(t) = -2e^{-t} + Ce^{1/4t}$$ Multiply both sides by $e^{1/4t}$

$$\Rightarrow V(0) = 0 \Rightarrow -2 + C = 0 \Rightarrow C = 2$$ Plug in $V(0) = ($ to solve for C

$$\Rightarrow V(t) = 2e^{1/4t} - 2e^{-t}$$

## Chapter 3: Linear Equations of Higher Order

(a) [20%] Solve for the general solution: $y'' + 6y' + 73y = 730$

A

The char. poly is $r^2 + 6r + 73 = 0$

$\Rightarrow r = \frac{-6 \pm \sqrt{36-292}}{2} = \frac{-6 \pm 16i}{2} = -3 \pm 8i$

$\Rightarrow$ Atoms are $e^{-3t}\cos 8t, e^{-3t}\sin 8t$

$\Rightarrow$ Solution to homogenous eq. is $y_h = C_1 e^{-3t}\cos 8t + C_2 e^{-3t}\sin 8t$

$\Rightarrow$ Method of un det. coeff. gives us the atom 1 for the particular sol.

$\Rightarrow y_p = C_3 \Rightarrow C_3'' + 6C_3' + 73C_3 = 730 \Rightarrow C_3 = 10$

$\Rightarrow y = y_h + y_p = C_1 e^{-3t}\cos 8t + C_2 e^{-3t}\sin 8t + 10$

(b) [30%] Given $5x''(t) + 2x'(t) + 10x(t) = F_0\cos(\omega t)$, which represents a damped forced spring-mass system with $m = 5$, $c = 2$, $k = 10$, answer questions (c1), (c2).

A **(c1)** Compute the frequency $\omega$ for practical mechanical resonance.

$$\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}} = \sqrt{2 - \frac{4}{50}} = \sqrt{\frac{96}{50}} = \boxed{\frac{4\sqrt{6}}{5\sqrt{2}}}$$

A **(c2)** Classify the homogeneous problem as over-damped (non-oscillatory), critically-damped or under-damped (oscillatory).

$b^2 = 2^2 = 4$

$4mk = 4(5)(10) = 200$

Therefore $b^2 < 4mk$ so

under damped

A (c) [20%] Define $y(x) = x\cos(3x) + 3x^3 e^x$. Construct the characteristic equation of a linear $n$th order homogeneous differential equation of least order $n$ which has $y(x)$ as a particular solution.

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The atoms are $\cos 3x, x\cos 3x, e^x, xe^x, x^2e^x, x^3e^x$
Therefore, the roots are $\pm 3i$ (mult. 2), 1 (mult. 4)
So the characteristic equation is $(r^2+9)^2(r-1)^4$

A (d) [30%] An $n$th order non-homogeneous differential equation is specified by its characteristic equation $(r+1)^3(r^2+100) = 0$ and the forcing term $f(x) = x^2 + x^2e^{-x} + xe^x + \sin(10x)$. Find the shortest trial solution for $y_p$ according to the method of undetermined coefficients. **Do not evaluate** undetermined coefficients.

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The roots of $(r+1)^3(r^2+100)$ are $-1$ (mult. 3), $\pm 10i$
$\Rightarrow$ Atoms are $e^{-x}, xe^{-x}, x^2e^{-x}, \cos 10x, \sin 10x$

Therefore, the atoms for $y_p$ are $x^2, x, 1, x^3e^{-x}, xe^x, e^x, x\sin 10x, x\cos 10x$

$\Rightarrow y_p = C_1x^2 + C_2x + C_3 + C_4x^3e^{-x} + C_5xe^x + C_6e^x + C_7x\sin 10x + C_8x\cos 10x$

## Chapters 4 and 5: Systems of Differential Equations

A (a) [20%] Let $A = \begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 5 \end{pmatrix}$, $\vec{\mathbf{v}}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{\mathbf{v}}_3 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$.

The eigenpairs of $A$ are $(-2, \vec{\mathbf{v}}_1), (2, \vec{\mathbf{v}}_2), (5, \vec{\mathbf{v}}_3)$.

A **(a1)** Apply the Eigenanalysis Method to solve $\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t)$.

The independent solutions are $ve^{\lambda t} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}e^{-2t}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}e^{5t}$

$\Rightarrow \vec{X}(t) = C_1\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}e^{-2t} + C_2\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}e^{2t} + C_3\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}e^{5t}$

A **(a2)** Show details for computing eigenpair $(2, \vec{\mathbf{v}}_2)$.

**Expected**: Show linear algebra details for computing $\vec{\mathbf{v}}_2$ for eigenvalue $\lambda = 2$. This involves row reduction plus display of the scalar solution and the vector solution.

$(A - \lambda I)v = 0 \Rightarrow \left(\begin{pmatrix} 0 & 2 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}\right)v = 0$

$\Rightarrow \begin{pmatrix} -2 & 2 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 3 \end{pmatrix}v = 0$

$\Rightarrow \begin{pmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}v = 0$ (sub row 3 from rows 1,2)

$\Rightarrow \begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}v = 0$ (add row 1 to row 2)

$\Rightarrow \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}v = 0$ (divide rows by const.)

$\Rightarrow v = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (Matrix is simple enough to inspect eqs. $v_3 = 0, -v_1 + v_2 = 0$), arbitrarily choose

**(b)** [20%] Find the scalar general solution of the $2 \times 2$ system

$$\begin{cases} x' = 7x + 2y, \\ y' = 2x + 7y \end{cases} \quad \text{or} \quad \frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 7 & 2 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

by the Cayley-Hamilton-Ziebur Method, using the textbook's Chapter 4 shortcut.

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$$\begin{vmatrix} 7-\lambda & 2 \\ 2 & 7-\lambda \end{vmatrix} = 0 \Rightarrow (7-\lambda)^2 - 4 = 0 \Rightarrow \lambda^2 - 14\lambda + 45 = 0$$

$$\Rightarrow (\lambda - 9)(\lambda - 5) = 0$$

$$\Rightarrow \lambda = 5, 9$$

The solution atoms are $e^{5t}, e^{9t}$

CHZ tells us $x = c_1 e^{5t} + c_2 e^{9t}$, $y = c_3 e^{5t} + c_4 e^{9t}$

$$\Rightarrow x' = 5c_1 e^{5t} + 9c_2 e^{9t} = 7x + 2y = (7c_1 + 2c_3)e^{5t} + (7c_2 + 2c_4)e^{9t}$$

$$\Rightarrow \begin{cases} 5c_1 = 7c_1 + 2c_3 \\ 9c_2 = 7c_2 + 2c_4 \end{cases} \Rightarrow \begin{cases} c_1 = -c_3 \\ c_2 = c_4 \end{cases}$$

$$\Rightarrow \begin{cases} x = c_1 e^{5t} + c_2 e^{9t} \\ y = -c_1 e^{5t} + c_2 e^{9t} \end{cases}$$ or alt, $\vec{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{5t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{9t}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c) [30%] Assume a $3 \times 3$ system $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}}$ has a vector general solution

$$\vec{\mathbf{u}}(t) = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -c_1 e^{5t} + c_2 e^{8t} \\ c_1 e^{5t} + c_2 e^{8t} \\ c_3 e^{t} \end{pmatrix}.$$

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**(c1)** Compute a $3 \times 3$ fundamental matrix $\Phi(t)$.

Three linearly ind. solutions are $\begin{pmatrix} -e^{5t} \\ e^{5t} \\ 0 \end{pmatrix}, \begin{pmatrix} e^{8t} \\ e^{8t} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ e^{t} \end{pmatrix}$

So a fundamental matrix $\Phi(t) = \begin{pmatrix} -e^{5t} & e^{8t} & 0 \\ e^{5t} & e^{8t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}$

**(c2)** Write a formula for the exponential matrix $e^{At}$ as an explicit matrix product. Do not multiply or simplify the product.

$$e^{At} = \Phi(t)\Phi(0)^{-1} = \Phi(t)\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -e^{5t} & e^{8t} & 0 \\ e^{5t} & e^{8t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}$$

**(c3)** Let $\vec{\mathbf{c}} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$. Display an explicit matrix-vector product for the solution $\vec{\mathbf{u}}(t)$ of the initial value problem $\frac{d}{dt}\vec{\mathbf{u}} = A\vec{\mathbf{u}}$, $\vec{\mathbf{u}}(0) = \vec{\mathbf{c}}$. Do not multiply or simplify the product.

$$\vec{u}(t) = e^{At}\vec{c} = \begin{pmatrix} -e^{5t} & e^{8t} & 0 \\ e^{5t} & e^{8t} & 0 \\ 0 & 0 & e^{t} \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

A

**(d)** [30%] Consider the $3 \times 3$ linear homogeneous system

$$\begin{cases} x' &= 6x - 2y \\ y' &= -2x + 6y, \\ z' &= y + z \end{cases} \qquad \text{or} \qquad \frac{d}{dt}\vec{\mathbf{u}}(t) = \begin{pmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 1 & 1 \end{pmatrix} \vec{\mathbf{u}}(t).$$

Solve the system by the most efficient method.

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$$\begin{pmatrix} 6-\lambda & -2 & 0 \\ -2 & 6-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\left(36 - 12\lambda + \lambda^2 - 4\right)$$

$$(1-\lambda)\ (\lambda^2 - 12\lambda + 32)$$

$$(1-\lambda)\ (\lambda - 8)(\lambda - 4)$$

$$\lambda = 1,\ 8,\ 4$$

$$\lambda = 1 \quad \begin{pmatrix} 6-1 & -2 & 0 \\ -2 & 6-1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 5 & -2 & 0 \\ 0 & 21 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \lambda = 1 \ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\lambda = 8 \quad \begin{pmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 1 & -7 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -7 \end{pmatrix}$$

$x_1 + x_2 = 0$

$x_2 + x_3 = 0$

$x_1 = -7$

$x_2 = 7$

$x_3 = 1$

$$\lambda = 8 \ \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix}$$

$$\lambda = 4 \quad \begin{pmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 1 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -3 \end{pmatrix} \qquad \lambda = 4 \ \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$$

$$\vec{u}(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2 e^{8t} \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$$

## Chapter 6: Dynamical Systems

Consider the nonlinear dynamical system

$$(1) \qquad \begin{cases} x' &= 16x - 4x^2 - xy, \\ y' &= 7y - y^2 - xy \end{cases}$$

A

**(a)** [20%] Find the four equilibrium points for nonlinear system (1). One of these is $x = 3, y = 4$.

$$\begin{cases} x' = 0 \\ y' = 0 \end{cases} \Rightarrow \begin{cases} 16x - 4x^2 - xy = 0 \\ 7y - y^2 - xy = 0 \end{cases} \Rightarrow \begin{cases} x(16 - 4x - y) = 0 \\ y(7 - y - x) = 0 \end{cases}$$

$\Rightarrow x = 0, y = 0$

$\Rightarrow x = 0, (7 - y - x) = 0 \Rightarrow x = 0, y = 7$

$\Rightarrow 16 - 4x - y = 0, y = 0 \Rightarrow x = 4, y = 0$

$\Rightarrow 16 - 4x - y = 0, 7 - y - x = 0 \Rightarrow x = 3, y = 4$

Therefore, the four equilibrium points are $(0,0), (0,7), (4,0), (3,4)$

A

**(b)** [20%] Compute the Jacobian matrix $J(x, y)$ for nonlinear system (1). Then evaluate $J(x, y)$ at equilibrium point $x = 3, y = 4$.

$$J(x,y) = \begin{pmatrix} dF/dx & dF/dy \\ dG/dx & dG/dy \end{pmatrix} = \begin{pmatrix} 16 - 8x - y & -x \\ -y & 7 - 2y - x \end{pmatrix}$$

$$\Rightarrow J(3,4) = \begin{pmatrix} 16 - 24 - 4 & -3 \\ -4 & 7 - 8 - 3 \end{pmatrix} = \begin{pmatrix} -12 & -3 \\ -4 & -4 \end{pmatrix}$$

**(c)** [30%] Consider nonlinear system (1). Classify the linearization at equilibrium point $x = 3, y = 4$ as a node, spiral, center, saddle. Do not sub-classify a node.

A

$$\begin{vmatrix} -12-\lambda & -3 \\ -4 & -4-\lambda \end{vmatrix} = 0 \Rightarrow (12+\lambda)(4+\lambda) - 12 = 0$$

$$\Rightarrow \lambda^2 + 16\lambda + 36 = 0$$

$$\Rightarrow \lambda = \frac{-16 \pm \sqrt{256-144}}{2} = -8 \pm \frac{\sqrt{112}}{2}$$

Since $\frac{\sqrt{112}}{2} < 8$, both $\lambda$ are negative

Therefore, the equil. point (3, 4) is a node

A

**(d)** [30%] Consider nonlinear system (1). Determine the possible classification of node, spiral, center or saddle and corresponding stability for equilibrium $x = 3, y = 4$ according to the **Pasting Theorem**, which is Theorem 2 in section 6.2 (Stability of Almost Linear Systems). State precisely the **two exceptions** of the pasting theorem, then explain how the theorem applies to nonlinear system (1) at $x = 3, y = 4$.

Since both eigenvalues are negative, the pasting theorem tells us the (3, 4) equil. point is a stable improper node

Ambiguity occurs when both eigenvalues are equal - the equilibrium point might be node or a spiral, or when both eigenvalues are pure imaginary - the equilibrium point might be a center or spiral

Since neither of the two ambiguous cases apply here, the pasting theorem guarantees our classification will be accurate after linearization.

# Chapter 7: Laplace Theory

Symbol $\delta(t)$ is the Dirac impulse. Symbol $u(t)$ is the unit step. Assumed below is experience with the following. Rules have precise hypotheses, omitted here for brevity.

Convolution Theorem. $\mathcal{L}(g_1)\mathcal{L}(g_2) = \mathcal{L}\left(\int_0^t g_1(t-x)g_2(x)dx\right)$

Periodic Function Theorem. $f(t+p) = f(t)$ implies $\mathcal{L}(f(t)) = \dfrac{\int_0^p f(t)dt}{1-e^{-ps}}$

Second Shifting Theorem Forward. $\mathcal{L}(g(t)u(t-a)) = e^{-as}\mathcal{L}\left(g(t)|_{t->t+a}\right)$

Second Shifting Theorem Backward. $e^{-as}\mathcal{L}(f(t)) = \mathcal{L}(f(t-a)u(t-a))$

Dirac Impulse Identity. $\int_0^\infty W(x)du(t-a) = W(a)$. Formally $\delta(t) = du(t)$. Then $\mathcal{L}(\delta(t-a)) = e^{-as}$ for $a \geq 0$.

Resolvent Identity. $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + \vec{\mathbf{F}}(t)$ has identity $(sI - A)\mathcal{L}(\vec{\mathbf{u}}) = \vec{\mathbf{u}}(0) + \mathcal{L}(\vec{\mathbf{F}})$.

Exponential Order. This means $|f(x)| \leq M\, e^{\alpha x}$ for some $M > 0$ and real number $\alpha$.

A

**(a)** [20%] Let $f(t)$ be continuous and of exponential order. Define $F(s) = \mathcal{L}(f(t))$. Prove the **Final Value Theorem**: $\lim_{s\to\infty} F(s) = 0$ (succinctly $F(\infty) = 0$).

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$F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

$\Rightarrow \lim_{s\to\infty} F(s) = \lim_{s\to\infty} \int_0^\infty e^{-st} f(t)\,dt$

$= \int_0^\infty 0\,dt$ (Since $f(t)$ is of exponential order, eventually $e^{-st}$ will overshadow $f(t)$).

$= 0$

Details needed

A

(b) [20%] Illustrate the convolution theorem by solving for $f(t)$ in the equation $\mathcal{L}(f(t)) = \frac{1}{s}\frac{1}{s+1}$. Check the answer with partial fractions.

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$$\mathcal{L}(f(t)) = \frac{1}{s}\frac{1}{s+1} \Rightarrow \mathcal{L}(f(t)) = \mathcal{L}(1)\mathcal{L}(e^{-t})$$

$$\Rightarrow f(t) = 1 * e^{-t}$$

$$= \int_0^t 1 \times e^{-\tau}\, d\tau$$

$$= -e^{-\tau}\Big|_0^t$$

$$= -e^{-t} + 1$$

By method of partial fractions, $\mathcal{L}(f(t)) = \frac{A}{s} + \frac{B}{s+1} = \frac{1}{s} - \frac{1}{s+1}$

$$\Rightarrow f(t) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1}\right)$$

$$= 1 - e^{-t} \ \checkmark$$

(c) [20%] Solve for $f(t)$ using the second shifting theorem: $\mathcal{L}(f(t)) = e^{-2s}\frac{1}{s+1}$.

A

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$$\mathcal{L}(f(t)) = e^{-2s}\frac{1}{s+1}$$

$$= e^{-2s}\mathcal{L}(e^{-t})$$

$$\Rightarrow f(t) = u(t-2)\, e^{-t+2}$$

**(d)** [20%] Symbol $\delta(t)$ is the Dirac impulse. Derive an expression for $\mathcal{L}(x(t))$ for the impulse problem

A

$$x''(t) + 100x(t) = 5\delta(t - \pi), \quad x(0) = 0, \quad x'(0) = 1.$$

To save time, do not solve for $x(t)$.

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$$x''(t) + 100x(t) = 5\delta(t-\pi)$$

$$\Rightarrow \mathcal{L}(x'' + 100x) = \mathcal{L}(5\delta(t-\pi))$$

$$\Rightarrow s^2\mathcal{L}(x) - 1 + 100\mathcal{L}(x) = 5e^{-\pi s}$$

$$\Rightarrow (s^2 + 100)\mathcal{L}(x) = 5e^{-\pi s} + 1$$

$$\Rightarrow \mathcal{L}(x) = \frac{5e^{-\pi s} + 1}{s^2 + 100}$$

$$\Rightarrow x = \mathcal{L}^{-1}\left(\frac{5e^{-\pi s} + 1}{s^2 + 100}\right)$$

(e) [20%] **Laplace Theory** applied to the forced linear dynamical system

$$\begin{cases} x' = \quad 4x - 2y + 2t, \\ y' = -2x + 4y, \\ x(0) = 0, y(0) = 0, \end{cases} \qquad \text{or} \qquad \begin{cases} \vec{\mathbf{u}}' = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \vec{\mathbf{u}} + \begin{pmatrix} 2t \\ 0 \end{pmatrix}, \\ \vec{\mathbf{u}}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{cases}$$

produces the formulas

$$\mathcal{L}(x(t)) = \frac{2s-8}{s^2(s-2)(s-6)}, \quad \mathcal{L}(y(t)) = \frac{-4}{s^2(s-2)(s-6)}.$$

A

Display the **Resolvent Method** solution steps that produce these formulas. To save time, **do not solve for** $x(t)$ **or** $y(t)$.

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$$\vec{u}' = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix} \vec{u} + \begin{pmatrix} 2t \\ 0 \end{pmatrix} \Rightarrow s\mathcal{L}(\vec{u}) - \vec{u}(0) = \mathcal{L}\left(\begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}\vec{u}\right) + \mathcal{L}\begin{pmatrix} 2t \\ 0 \end{pmatrix}$$

$$\Rightarrow s\mathcal{L}(\vec{u}) - \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}\mathcal{L}(\vec{u}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mathcal{L}\begin{pmatrix} 2t \\ 0 \end{pmatrix}$$

$$\Rightarrow (sI - A)\mathcal{L}(\vec{u}) = \begin{pmatrix} 2/s^2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \mathcal{L}(\vec{u}) = \begin{pmatrix} s-4 & 2 \\ 2 & s-4 \end{pmatrix}^{-1} \begin{pmatrix} 2/s^2 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} s-4 & 2 \\ 2 & s-4 \end{pmatrix} = (s-4)^2 - 4 = s^2 - 8s + 12 = (s-2)(s-6)$$

$$\Rightarrow \mathcal{L}(\vec{u}) = \frac{1}{(s-2)(s-6)} \begin{pmatrix} s-4 & -2 \\ -2 & s-4 \end{pmatrix} \begin{pmatrix} 2/s^2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \mathcal{L}(\vec{u}) = \begin{pmatrix} 2s-8/s^2(s-2)(s-6) \\ -4/s^2(s-2)(s-6) \end{pmatrix}$$

$$\Rightarrow \begin{cases} \mathcal{L}(x(t)) = \dfrac{2s-8}{s^2(s-2)(s-6)} \\ \mathcal{L}(y(t)) = \dfrac{-4}{s^2(s-2)(s-6)} \end{cases}$$

## Chapter 9: Fourier Series and Partial Differential Equations

In part (a), let $f_0(x) = 2$ on the interval $1 < x < 2$, $f_0(x) = 0$ for all other values of $x$ on $-2 \le x \le 2$. Let $f(x)$ be the periodic extension of $f_0$ to the whole real line, of period 4. The Fourier series of $f(x)$ on $|x| \le L$ is

$$a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x/L) + b_n \sin(n\pi x/L).$$

Formulas exist for $a_n$, $b_n$ expressed in terms of $f$, using inner product spaces.

**(a)** [20%] Compute the Fourier coefficients $a_5$ and $b_5$ of $f(x)$ on $[-2, 2]$. Warning: $f$ is neither even nor odd.

A

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{1}^{2} 2\cos\frac{n\pi x}{2}\,dx = \frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_1^2$$

$$\Rightarrow a_5 = \frac{2}{5\pi}\sin\frac{5\pi x}{2}\Big|_1^2$$

$$= \frac{2}{5\pi}\left(\sin(5\pi) - \sin\left(\frac{5\pi}{2}\right)\right) = -\frac{2}{5\pi}$$

$$b_5 = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{5\pi x}{2}\,dx = \frac{1}{2}\int_{1}^{2} 2\sin\frac{5\pi x}{2} = \frac{-2}{5\pi}\cos\frac{5\pi x}{2}\Big|_1^2$$

$$= \frac{-2}{5\pi}\left(\cos 5\pi - \cos\frac{5\pi}{2}\right)$$

$$= \frac{-2}{5\pi}(-1-0)$$

$$= 2/5\pi$$

In part (b), let $g_0(x) = 1$ on the interval $-2 < x < 0$, $g_0(x) = 2$ on the interval $0 < x < 2$, $g_0(x) = 0$ for all other values of $x$ on $-2 \leq x \leq 2$. Let $g(x)$ be the periodic extension of $g_0$ to the whole real line, of period 4.

A

**(b)** [10%] Find all values of $x$ in $-3 < x < 5$ for which the Fourier series of $g$ will exhibit Gibb's over-shoot.

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Gibb's over-shoot occurs whenever there is a jump discontinuity. In this case, it will occur at $x = -2, 0, 2, 4$

**(c)** [10%] Assume $h(x)$ is a piecewise continuous function on $(-\infty, \infty)$. Let $H(x)$ be the Fourier series of $h(x)$ on $-L \leq x \leq L$. Does $H(0) = h(0)$ hold no matter the choice of $h$? Cite a theorem or invent a counterexample.

A

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No, let $h(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$, Clearly, $h$ is piecewise smooth since all derivatives are 0

Therefore, by the Fourier convergence theorem, $H(0) = \left(\lim_{x \to 0^+} h(x) + \lim_{x \to 0^-} h(x)\right)/2$

$$= (-1+1)/2$$
$$= 0$$

which is not equal to $h(0) = 1$.

**(d) [30%] Heat Conduction in a Rod.**
Let $L = 2$ (rod length), $k = 1$ (conduction constant). Solve the rod problem on $0 \le x \le L$, $t \ge 0$:

$$\begin{cases} u_t &= k\,u_{xx}, \\ u(0,t) &= 0, \\ u(L,t) &= 0, \\ u(x,0) &= \sum_{n=1}^{\infty} \frac{1}{n+1} \sin(2n\pi x) \end{cases}$$

---

A

Let $U = X(x)T(t)$

$\Rightarrow XT' = kX''T$

$\Rightarrow T'/kT = X''/X = -\lambda$

$\Rightarrow \begin{cases} X'' + \lambda X = 0 \\ T' + k\lambda T = 0 \end{cases}$

$\Rightarrow r^2 + \lambda = 0 \Rightarrow r = \pm\sqrt{\lambda}$

$\Rightarrow \lambda < 0$ necessarily to satisify $U(0,t) = U(L,t) = 0$

$\Rightarrow X = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$

$= B\sin\sqrt{\lambda}x$ since $U(0,t) = 0$

$\Rightarrow \lambda = \frac{n^2\pi^2}{4}$ since $U(L,t) = 0$

$\Rightarrow X = B\sin\frac{n\pi x}{2}$

$\Rightarrow T' + \frac{n^2\pi^2}{4}T = 0 \Rightarrow T = e^{-\frac{n^2\pi^2 t}{4}}$ by int. fact

$\Rightarrow U_n = X_n T_n = A_n e^{-\frac{n^2\pi^2 t}{4}} \sin\frac{n\pi x}{2}$ (combine constants)

$\Rightarrow U = \sum U_n = \sum A_n e^{-\frac{n^2\pi^2 t}{4}} \sin\frac{n\pi x}{2}$

$\Rightarrow U = \sum_{n=1}^{\infty} \frac{1}{n+1} e^{-4n^2\pi^2 t} \sin 2n\pi x$ (equate coeffs. and sub $n = 4n$)

**DEFINITION**. The **normal modes** for the string equation $u_{tt} = c^2 u_{xx}$ on $0 < x < L$, $t > 0$ are given by the functions

$$\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right), \quad \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right).$$

Each normal mode is a solution of the string equation. A superposition of normal modes is also a solution of the string equation.

**(e)** [30%] **Vibration of a Finite String.**
Let $L = 4$ (string length), $c = 4$ (wave speed). Solve the finite string vibration problem on $0 \le x \le L$, $t > 0$:

$$\begin{cases} u_{tt}(x,t) &= c^2 u_{xx}(x,t), \\ u(0,t) &= 0, \\ u(4,t) &= 0, \\ u(x,0) &= \sin(5\pi x) + 5\sin(7\pi x), \\ u_t(x,0) &= \sin(7\pi x) + 12\sin(15\pi x). \end{cases}$$

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$C = 4 \quad L = 4$

$u(x,t) = X(x)\,T(t) \qquad X(x) = \sin\left(\frac{n\pi x}{4}\right)$

$T(t) = c_n \cos\left(n\pi x\right) + d_n \sin\left(n\pi x\right)$

$u(x,0) = \sum_{n=1}^{\infty} X(x)\,T(0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{4}\right) = \sin(5\pi x) + 5\sin(7\pi x)$

$c_n = 0 \ \forall n$ except $c_{20} = 1$, $c_{28} = 5$

$u_t(x,0) = \sum_{n=1}^{\infty} X(x)\,T'(0) = \sum_{n=1}^{\infty} d_n\, n\pi \sin\left(\frac{n\pi x}{4}\right) = \sin(7\pi x) + 12\sin(15\pi x)$

$n = 28 \qquad 28\, d_{28}\, \pi = 1 \Rightarrow d_{28} = \frac{1}{21\cdot\pi} \qquad d_n = 0$

$n = 60 \qquad 60\, \pi\, d_{45} = 1 \Rightarrow d_{60} = \frac{1}{45\pi} \qquad$ except $d_{28} = \frac{1}{28\pi}$

$d_{60} = \frac{1}{460\pi}$

$u(x,t) = \sin(5\pi x)\cos(15\pi t) + 5\sin(7\pi x)\cos(21\pi t) +$

$\frac{1}{21\pi}\sin(7\pi x)\sin(21\pi t) + \frac{12}{45\pi}\sin(15\pi x)\sin(45\pi t)$

used $c=3$ $\qquad$ $c=3$