$\cosh(at) = \frac{s}{s^2-a^2}$ $e^{at} = \frac{1}{s-a}$

$\cos(at) = \frac{s}{s^2+a^2}$ $t^n = \frac{n!}{s^{n+1}}$

$\sin(at) = \frac{a}{s^2+a^2}$ $\sinh(at) = \frac{a}{s^2-a^2}$

Differential Equations 2280

Midterm Exam 2

Exam Date: 5 April 2019 at 7:30am

Solution KEY

High Scores = $\frac{587}{600}$, $\frac{586}{600}$

Instructions: This in-class exam is 80 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

Laplace Theory: Chapter 7

Laplace Theory Background. Expected without notes or books is the 4-entry Forward Laplace Table and these rules: Linearity, Parts, Shift, s-differentiation, Lerch's Theorem, Final Value Theorem, Existence theory.

1. **(Laplace Theory: Differential Equations)**
   Solve differential equations (1a), (1b) by Laplace's method.

A

(1a) [40%] Third order $x''' + x' = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$.

$$\mathcal{L}(x''') + \mathcal{L}(x') = 0$$

$$s\mathcal{L}(x'') - 0 + s\mathcal{L}(x) - 1 = 0$$

$$s^2\mathcal{L}(x') - 0 + s\mathcal{L}(x) - 1 = 0$$

$$s^3\mathcal{L}(x) - s^2 + s\mathcal{L}(x) - 1 = 0$$

$$(s^3 + s)\mathcal{L}(x) = s^2 + 1$$

$$\mathcal{L}(x) = \frac{s^2+1}{s^3+s}$$

$$\mathcal{L}(x) = \frac{s^2+1}{s(s^2+1)}$$

$$\mathcal{L}(x) = \frac{1}{s}$$

$$\mathcal{L}(x) = \mathcal{L}(1)$$

$$\boxed{x = 1}$$

(1b) [60%] Dynamical system $x' = x + y$, $y' = 2x + 6$, $x(0) = 0$, $y(0) = 0$.

A

$$\mathcal{L}(x') = \mathcal{L}(x) + \mathcal{L}(y)$$
$$s\mathcal{L}(x) - 0 = \mathcal{L}(x) + \mathcal{L}(y)$$
$$(s-1)\mathcal{L}(x) = \mathcal{L}(y)$$
$$\mathcal{L}(y) = \frac{6(s-1)}{s(s-2)(s+1)}$$
$$\mathcal{L}(y) = \frac{6s-6}{s(s-2)(s+1)} = \frac{6s}{s(s-2)(s+1)} - \frac{6}{s(s-2)(s+1)}$$
$$= \frac{6}{(s-2)(s+1)} - \frac{6}{s(s-2)(s+1)}$$
$$\mathcal{L}(y) = \frac{a}{s-2} + \frac{b}{s+1} - \left(\frac{c}{s} + \frac{d}{s-2} + \frac{f}{s+1}\right)$$
$$\mathcal{L}(y) = \mathcal{L}(ae^{2t}) + \mathcal{L}(be^{-t}) - \mathcal{L}(c) - \mathcal{L}(de^{2t}) - \mathcal{L}(fe^{-t})$$
$$y = ae^{2t} + be^{-t} - c - de^{2t} - fe^{-t}$$
$$y = (a-d)e^{2t} + (b-f)e^{-t} - c$$
$$\boxed{y = 3e^{2t} - 4e^{-t} - 3}$$

$$\frac{6}{(s-2)(s+1)} = \frac{a}{s-2} + \frac{b}{s+1}$$
$$6 = a(s+1) + b(s-2)$$
$$a = 2 \qquad b = -2$$

$$\frac{-6}{s(s-2)(s+1)} = \frac{c}{s} + \frac{d}{s-2} + \frac{f}{s+1}$$
$$-6 = c(s-2)(s+1) + d(s)(s+1) + f(s)(s-2)$$
$$c = 3 \qquad d = -1 \qquad f = -2$$

$$\mathcal{L}(y') = 2\mathcal{L}(x) + \mathcal{L}(6)$$
$$s\mathcal{L}(y) = 2\mathcal{L}(x) + \frac{6}{s}$$
$$s(s-1)\mathcal{L}(x) = 2\mathcal{L}(x) + \frac{6}{s}$$
$$s(s-1)\mathcal{L}(x) - 2\mathcal{L}(x) = \frac{6}{s}$$
$$(s^2 - s - 2)\mathcal{L}(x) = \frac{6}{s}$$
$$(s-2)(s+1)\mathcal{L}(x) = \frac{6}{s}$$
$$\mathcal{L}(x) = \frac{6}{s(s-2)(s+1)}$$
$$\mathcal{L}(x) = \frac{6}{s(s-2)(s+1)} = \frac{A}{s} + \frac{B}{s-2} + \frac{C}{s+1}$$
$$\mathcal{L}(x) = \mathcal{L}(A) + \mathcal{L}(Be^{2t}) + \mathcal{L}(Ce^{-t})$$
$$x = A + Be^{2t} + Ce^{-t}$$
$$\boxed{x = -3 + e^{2t} + 2e^{-t}}$$

$$6 = A(s-2)(s+1) + B(s)(s+1) + C(s)(s-2)$$
$$A = -3 \qquad C = 2 \qquad B = 1$$

2. (Laplace Theory: Backward Table)
Solve for $f(t)$ in (2a), (2b), (2c), (2d).

Assumptions. Below, $f(t)$ is of piecewise continuous of exponential order. Expression $u(t)$ denotes the unit step function.

Credit. Document all steps, e.g., if you cancel $\mathcal{L}$ then cite Lerch's Theorem. The answer is 25%. The documented steps are 75%. Partial fraction coefficients are expected to be evaluated last.

(2a) [20%] $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+25)(s^2+4)}$ $= \dfrac{A}{s^2+25} + \dfrac{B}{s^2+4}$

A

$\mathcal{L}(f(t)) = \mathcal{L}\left(\frac{1}{5}A\sin(5t)\right) + \mathcal{L}\left(\frac{1}{2}B\sin(2t)\right)$

$f(t) = \frac{1}{5}A\sin(5t) + \frac{1}{2}B\sin(2t)$ Lerch's Thm

$f(t) = \frac{1}{5}\left(-\frac{100}{21}\right)\sin(5t) + \frac{1}{2}\left(\frac{100}{21}\right)\sin(2t)$

$\boxed{f(t) = -\frac{20}{21}\sin(5t) + \frac{50}{21}\sin(2t)}$

$100 = A(s^2+4) + B(s^2+25)$

$(100 = 4A + 25B)\,{-5}$

$(100 = 5A + 26B)\,4$

$-500 = -20A - 125B$

$400 = 20A + 104B$

$-100 = -21B$

$B = \frac{100}{21}$

$100 = 4A + 25\left(\frac{100}{21}\right)$

$100 - \frac{25(100)}{21} = 4A$

$\frac{2100}{21} - \frac{2500}{21} = 4A$

$-\frac{400}{21} = 4A$

$A = -\frac{100}{21}$

$\frac{100}{104}$

(2b) [20%] $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2+4s+29}$.

A

$\mathcal{L}(f(t)) = \dfrac{s+1}{(s+2)^2+25} = \dfrac{(s+2)-1}{(s+2)^2+25} = \dfrac{(s+2)}{(s+2)^2+25} - \dfrac{1}{(s+2)^2+25}$

$\mathcal{L}(f(t)) = \mathcal{L}(e^{-2t}\cos(5t)) + \mathcal{L}\left(-\frac{1}{5}e^{-2t}\sin(5t)\right)$ shifting thm $|_{s\to s+2}$

$\boxed{f(t) = e^{-2t}\cos(5t) - \frac{1}{5}e^{-2t}\sin(5t)}$ Lerch's Thm

(2c) [30%] $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+2s)(s^2-3s)}$ $= \dfrac{1}{s(s+2)s(s-3)} = \dfrac{1}{s^2(s+2)(s-3)}$

A

$(s+2)(s-3)$
s^2-s-6

$$\mathcal{L}(f(t)) = \frac{1}{s^2(s+2)(s-3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s-3}$$

$$\mathcal{L}(f(t)) = \mathcal{L}(A) + \mathcal{L}(Bt) + \mathcal{L}(Ce^{-2t}) + \mathcal{L}(De^{3t})$$

$$f(t) = A + Bt + Ce^{-2t} + De^{3t}$$ Lerch's Thm

$$\boxed{f(t) = \frac{1}{36} - \frac{1}{6}t - \frac{1}{20}e^{-2t} + \frac{1}{45}e^{3t}}$$

$$1 = A(s)(s+2)(s-3) + B(s+2)(s-3) + C(s^2)(s-3) + D(s^2)(s+2)$$

$$B = -\frac{1}{6} \qquad C = -\frac{1}{20} \qquad D = \frac{1}{45}$$

$$\cancel{1} = As^3 - As^2 - 6As - \frac{1}{6}s^2 + \frac{1}{6}s + \cancel{1} - \frac{1}{20}s^3 + \frac{3}{20}s^2 + \frac{1}{45}s^3 + \frac{2}{45}s^2$$

$$0 = \left(A - \frac{1}{20} + \frac{1}{45}\right)s^3 + \left(-A - \frac{1}{6} + \frac{3}{20} + \frac{2}{45}\right)s^2 + \left(-6A + \frac{1}{6}\right)s$$

$$-6A + \frac{1}{6} = 0$$

$$A = \frac{1}{36}$$

(2d) [30%] $\dfrac{d}{ds}\mathcal{L}(f(t))\Big|_{s\to(s-3)} = \dfrac{d}{ds}\mathcal{L}(u(t-\pi)e^t\cos 25t).$

A

$$\mathcal{L}((-t)f(t))\Big|_{s\to s-3} = \mathcal{L}((-t)u(t-\pi)e^t(\cos 25t))$$ s-differential thm

$$\mathcal{L}((-t)e^{3t}f(t)) = \mathcal{L}(-t\,u(t-\pi)e^t\cos 25t)$$ shifting thm

$$-te^{3t}f(t) = -t\,u(t-\pi)e^t\cos 25t$$ Lerch's thm

$$f(t) = \frac{u(t-\pi)e^t\cos(25t)}{e^{3t}}$$

$$\boxed{f(t) = u(t-\pi)e^{-2t}\cos(25t)}$$

3. (Laplace Theory: Forward Table)
Compute the Laplace transform $\mathcal{L}(f(t))$.

(3a) [20%] $f(t) = (-t)e^{2t}\sin(3t)$.

A

$\mathcal{L}(f(t)) = \mathcal{L}((-t)\, e^{2t}\sin(3t))$

$= \frac{d}{ds}\mathcal{L}(e^{2t}\sin 3t)$ s-differential thm

$= \frac{d}{ds}\mathcal{L}(\sin 3t)\Big|_{s\to s-2}$ → shifting thm

$= \frac{d}{ds}\left(\frac{3}{s^2+9}\Big|_{s\to s-2}\right)$

$= \frac{d}{ds}\frac{3}{(s-2)^2+9}$

$\mathcal{L}(f(t)) = \frac{d}{ds}\, 3\left((s-2)^2+9\right)^{-1}$

$= -3\left((s-2)^2+9\right)^{-2}\cdot 2(s-2)$

$$\boxed{\mathcal{L}(f(t)) = \frac{-6(s-2)}{((s-2)^2+9)^2}}$$

(3b) [30%] $f(t)) = e^{-\pi t}g(t)$ and $g(t) = \dfrac{e^{2t} - e^{-2t}}{t}$.

A

$\mathcal{L}(t\, g(t)) = \mathcal{L}(e^{2t} - e^{-2t})$

$-\frac{d}{ds}\mathcal{L}(g(t)) = \mathcal{L}(e^{2t} - e^{-2t})$ s-differential thm

$-\frac{d}{ds}\mathcal{L}(g(t)) = \frac{1}{s-2} - \frac{1}{s+2}$

$\frac{d}{ds}\mathcal{L}(g(t)) = \frac{1}{s+2} - \frac{1}{s-2}$

$\mathcal{L}(g(t)) = \ln|s+2| - \ln|s-2|$

$\mathcal{L}(g(t)) = \ln\left|\frac{s+2}{s-2}\right|$

$\mathcal{L}(e^{-\pi t}g(t)) = \mathcal{L}(g(t))\Big|_{s\to s+\pi}$ shifting thm

$$\boxed{\mathcal{L}(e^{\pi t}g(t)) = \ln\left|\frac{s+\pi+2}{s+\pi-2}\right|}$$

(3c) [20%] $\frac{d}{ds}\mathcal{L}(f(t)) = \frac{1}{s^{2019}} + \frac{1}{s^2+1}$.

A -

$\mathcal{L}((-t)f(t)) = \frac{1}{s^{2019}} + \frac{1}{s^2+1}$ s-differential thm

$\mathcal{L}((-t)f(t)) = \mathcal{L}\left(\frac{1}{2018!}t^{2018}\right) + \mathcal{L}(\sin t)$ yes, if finding $f(t)$

$(-t)f(t) = \frac{t^{2018}}{2018!} + \sin t$ Lerch's thm

$$\boxed{f(t) = \frac{t^{2018}}{(-t)\,2018!} + \frac{1}{-t}\sin t}$$

← A common error to find $f(t)$ instead of $\mathcal{L}(f)$

Find $\mathcal{L}(f) \rightarrow$ $\frac{d}{ds}\mathcal{L}(f(t)) = \frac{1}{s^{2019}} + \frac{1}{s^2+1}$

$\mathcal{L}(f(t)) = \int \frac{1}{s^{2019}} + \frac{1}{s^2+1}\,ds$ = $\frac{s^{-2018}}{-2018} + \arctan(s) + c$

$\mathcal{L}(f(t)) = \int \mathcal{L}\left(\frac{1}{2018!}t^{2018} + \sin t\right) ds$ $c=0$ by final Value Thm.

$\mathcal{L}(f(t)) = \frac{1}{s}\mathcal{L}\left(\frac{1}{2018!}t^{2018} + \sin t\right)$ NO s-diff thm

$$\boxed{\mathcal{L}(f(t)) = \frac{1}{s^{2020}} + \frac{1}{s(s^2+1)}}$$

(3d) [30%] Define $f(t) = e^{-2t}g(t))$ where $g(t) = \frac{e^{2t} - e^{-2t}}{t}$.

A

$\mathcal{L}(t\,g(t)) = \mathcal{L}(e^{2t} - e^{-2t})$

$-\frac{d}{ds}\mathcal{L}(g(t)) = \frac{1}{s-2} - \frac{1}{s+2}$ s-differential thm

$\frac{d}{ds}\mathcal{L}(g(t)) = \frac{1}{s+2} - \frac{1}{s-2}$ ok

$\mathcal{L}(g(t)) = \ln|s+2| - \ln|s-2|$ $+c$ $c=0$ by final Value Thm

$\mathcal{L}(g(t)) = \ln\left|\frac{s+2}{s-2}\right|$

$\mathcal{L}(f(t)) = \mathcal{L}(g(t))\Big|_{s \rightarrow s+2}$ shifting thm good

$$\boxed{\mathcal{L}(f(t)) = \ln\left|\frac{s+4}{s}\right|}$$

Systems of Differential Equations: Chapters 4 and 5

Background.

The **Eigenanalysis Method** for a real 3×3 matrix A assumes eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$, $(\lambda_3, \vec{\mathbf{v}}_3)$. It says that the 3×3 system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has general solution $\vec{\mathbf{x}}(t) = c_1 \vec{\mathbf{v}}_1 e^{\lambda_1 t} + c_2 \vec{\mathbf{v}}_2 e^{\lambda_2 t} + c_3 \vec{\mathbf{v}}_3 e^{\lambda_3 t}$.

The **Cayley-Hamilton-Ziebur method** is based upon this result:

> Let A be an $n \times n$ real matrix. The components of solution $\vec{\mathbf{u}}$ of $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$. Alternatively, $\vec{\mathbf{u}}(t)$ is a vector linear combination of the Euler solution atoms: $\vec{\mathbf{u}}(t) = \sum_{k=1}^{n}(atom_k)\vec{\mathbf{d}}_k$.

A **Fundamental Matrix** is an $n \times n$ matrix $\Phi(t)$ with columns consisting of independent solutions of $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$, where A is an $n \times n$ real matrix. The general solution of $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ is $\vec{\mathbf{x}}(t) = \Phi(t)\vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, with $|\Phi(0)| \neq 0$ required to establish independence of the columns of Φ.

The **Exponential Matrix**, denoted e^{At}, is the unique fundamental matrix $\Psi(t)$ such that $\Psi(0) = I$. Matrix A is an $n \times n$ real matrix. It is known that $e^{At} = \Phi(t)\Phi(0)^{-1}$ for *any* fundamental matrix $\Phi(t)$. Consequently, $\frac{d}{dt}\left(e^{At}\right) = A\,e^{At}$ and $e^{At}\Big|_{t=0} = I$.

4. (Systems: Eigenanalysis Method)
Complete parts ((**4a**), (**4b**), (**4c**).

(4a) [40%] Let $A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{pmatrix}$. Display the linear algebra details for computing the three eigenpairs of A.

A

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 0 & 0 & 7-\lambda \end{vmatrix} = (7-\lambda)((5-\lambda)(5-\lambda) - 1) = 0$$

$$(7-\lambda)(25-10\lambda+\lambda^2) - 7+\lambda = 0$$

$$175 - 70\lambda + 7\lambda^2 - 25\lambda + 10\lambda^2 - \lambda^3 - 7 + \lambda = 0$$

$$-\lambda^3 + 17\lambda^2 - 94\lambda + 168 = 0$$

$$\lambda^3 - 17\lambda^2 + 19\lambda - 168 = 0$$

$$(\lambda-6)(\lambda-7)(\lambda-4) = 0$$

$\lambda = 6, 7, 4$ → eigenvalues

$$\begin{array}{r} {}^{3}25 \\ \times 7 \\ \hline 175 \\ -7 \\ \hline 168 \end{array}$$

$(A-6I)v_1 = \vec{0}$ $\quad \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{cases} x_1 = t_1 \\ x_2 = t_1 \\ x_3 = 0 \end{cases}$ $\quad v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$(A-7I)v_2 = \vec{0}$ $\quad \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -2 & 1 & 0 \\ 3 & -3 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_2 = $

$\begin{cases} x_1 = 0 \\ x_2 = 0 \\ x_3 = t_1 \end{cases}$ $\quad v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$(A-4\lambda)v_3 = \vec{0}$ $\quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} v_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\begin{cases} x_1 = -t_1 \\ x_2 = t_1 \\ x_3 = 0 \end{cases}$ $\quad v_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

eigenpairs: $\quad 6, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad 7, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad 4, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

(4b) [30%] Matrix $A = \begin{pmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{pmatrix}$ has eigenpairs $\left(2, \begin{pmatrix} 1 \\ -5 \\ -3 \\ 3 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$, $\left(4, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$, $\left(5, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$. Display the general solution of $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$.

A

$$u(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ -5 \\ -3 \\ 3 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_3 e^{4t}\begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c_4 e^{5t}\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

(4c) [40%] Let $\vec{\mathbf{u}}(t) = \vec{\mathbf{c}}\,e^{rt}$, the vector Euler substitution. Assume real $n \times n$ matrix A has a real eigenpair $(\lambda, \vec{\mathbf{v}})$. Prove that the Euler substitution with $r = \lambda$ and $\vec{\mathbf{c}} = \vec{\mathbf{v}}$ applied to $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ finds a nonzero solution of $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$.

A

$$\frac{d}{dt}(\vec{c}e^{rt}) = A(\vec{c}e^{rt})$$

$$\vec{c}re^{rt} = A\vec{c}e^{rt}$$

$$\vec{c}r = A\vec{c}$$

$$A\vec{c} - \vec{c}rI = 0$$

$$(A - rI)\vec{c} = 0$$

when $r = \lambda$ and $\vec{c} = \vec{v}$

$(A - \lambda I)\vec{v} = 0 \Rightarrow$ a nonzero solution of $\vec{u}'(t) = A\vec{u}(t)$

5. (Systems: First Order Cayley-Hamilton-Ziebur)

(5a) [30%] The eigenvalues are 2, 6 for the matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$.

Display the general solution of $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5).

$$\boxed{x(t) = c_1 e^{2t} + c_2 e^{6t}}$$

$$x' = 4x + 2y$$
$$y' = 2x + 4y$$

$$x'(t) = 2c_1 e^{2t} + 6c_2 e^{6t}$$
$$2c_1 e^{2t} + 6c_2 e^{6t} = 4c_1 e^{2t} + 4c_2 e^{6t} + 2y$$
$$-2c_1 e^{2t} + 2c_2 e^{6t} = 2y$$

$$\boxed{y = -c_1 e^{2t} + c_2 e^{6t}}$$

(5b) [40%] The 3×3 triangular matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{pmatrix}$ represents a linear cascade, such as found in brine tank models. Apply the **linear integrating factor method** and related shortcuts (Ch 1 in the textbook) to find the components x_1, x_2, x_3 of the vector general solution $\vec{\mathbf{x}}(t)$ of $\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t)$.

A

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t)$$

$$\Rightarrow \begin{cases} x_1' = 2x_1 + x_2 + x_3 & (1) \\ x_2' = 5x_2 + x_3 & (2) \\ x_3' = -2x_3 & (3) \end{cases}$$ Rewrite as system of eq.s

$\Rightarrow x_3' + 2x_3 = 0$ Rearrange (3)

✓ $\Rightarrow x_3 = C_3 e^{-2t}$ Shortcut for x_h

$\Rightarrow x_2' - 5x_2 = C_3 e^{-2t}$ Rearrange (2) and sub. x_3

$\Rightarrow W = e^{\int 5\,dt} = e^{-5t}$ Int. factor formula

$\Rightarrow e^{-5t} x_2 = \int C_3 e^{-7t}\,dt$ Int. factor method

$\Rightarrow e^{-5t} x_2 = \frac{-C_3 e^{-7t}}{7} + C_2$ Evaluate RHS integral

✓ $\Rightarrow x_2 = C_2 e^{5t} - \frac{C_3 e^{-2t}}{7}$ Algebra

$\Rightarrow x_1' - 2x_1 = C_2 e^{5t} - \frac{C_3 e^{-2t}}{7} + C_3 e^{-2t} = C_2 e^{5t} + \frac{6}{7} C_3 e^{-2t}$ Rearrange (1) and sub x_2

$\Rightarrow W = e^{\int -2\,dt} = e^{-2t}$ Int. factor formula

$$\Rightarrow e^{-2t} x_1 = \int C_2 e^{3t} + \frac{6}{7} C_3 e^{-4t}\,dt$$
$$= \frac{C_2 e^{3t}}{3} - \frac{3}{14} C_3 e^{-4t} + C_1$$ Int. factor method

✓ $$\Rightarrow \begin{cases} x_1 = C_1 e^{2t} + \frac{1}{3} C_2 e^{5t} - \frac{3}{14} C_3 e^{-2t} \\ x_2 = C_2 e^{5t} + \frac{6}{7} C_3 e^{-2t} \\ x_3 = C_3 e^{-2t} \end{cases}$$ Algebra

ok above. Bad copy

(5c) [30%] The Cayley-Hamilton-Ziebur shortcut applies to the system

$$x' = 3x + 2y, \quad y' = -2x + 3y,$$

which has complex eigenvalues $\lambda = 3 \pm 2i$. Find a fundamental matrix $\Phi(t)$ for this system, documenting all details of the computation.

A

$$x = c_1 e^{3t}\cos(2t) + c_2 e^{3t}\sin(2t)$$

$$x' = 3c_1e^{3t}\cos(2t) - 2c_1e^{3t}\sin(2t) + 3c_2e^{3t}\sin(2t) + 2c_2e^{3t}\cos(2t)$$

$$3c_1e^{3t}\cos(2t) - 2c_1e^{3t}\sin(2t) + 3c_2e^{3t}\sin(2t) + 2c_2e^{3t}\cos(2t) =$$

$$=$$

$$3(c_1e^{3t}\cos(2t) + c_2e^{3t}\sin(2t)) + 2y$$

$$\cancel{3c_1e^{3t}\cos(2t) - 3c_1e^{3t}\cos(2t)} - 2c_1e^{3t}\sin(2t) + \cancel{3c_2e^{3t}\sin(2t) - 3c_2e^{3t}\sin(2t)} + 2c_2e^{3t}\cos(2t) = 2y$$

$$-2c_1e^{3t}\sin(2t) + 2c_2e^{3t}\cos(2t) = 2y$$

$$y = -c_1e^{3t}\sin(2t) + c_2e^{3t}\cos(2t)$$

$$\boxed{\phi(t) = \begin{bmatrix} e^{3t}\cos(2t) & e^{3t}\sin(2t) \\ -e^{3t}\sin(2t) & e^{3t}\cos(2t) \end{bmatrix}}$$

$$\frac{\partial}{\partial c_1} = \begin{bmatrix} e^{3t}\cos(2t) \\ -e^{3t}\sin(2t) \end{bmatrix} \qquad \frac{\partial}{\partial c_2} = \begin{bmatrix} e^{3t}\sin(2t) \\ e^{3t}\cos(2t) \end{bmatrix}$$

6. (Systems: Second Order Cayley-Hamilton-Ziebur)

Assume below that real 2×2 matrix $A = \begin{pmatrix} -13 & -6 \\ 6 & -28 \end{pmatrix}$ has eigenpairs $\left(-25, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$, $\left(-16, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$. Textbook theorems applied to $\frac{d^2}{dt^2}\vec{u}(t) = A\vec{u}(t)$ report general solution

$$\vec{u}(t) = (c_1 \cos(5t) + c_2 \sin(5t)) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c_3 \cos(4t) + c_4 \sin(4t)) \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(6a) [30%] Derive the characteristic equation $|A - r^2 I| = 0$ for the second order equation $\frac{d^2}{dt^2}\vec{u}(t) = A\vec{u}(t)$ from Euler's vector substitution $\vec{u}(t) = \vec{c}\, e^{rt}$. A proof is expected with details.[1]

A

$\frac{d^2}{dt^2}\vec{u}(t) = \frac{d^2}{dt^2}\vec{c}e^{rt}$ by subst.

$= \frac{d}{dt} r\vec{c}e^{rt} = r^2\vec{c}e^{rt}$

$r^2\vec{c}e^{rt} = A\vec{c}e^{rt}$ by subst (LHS = RHS)

$r^2 I e^{rt} = A\vec{c}e^{rt}$

$A\vec{c}e^{rt} = r^2 I\vec{c}e^{rt}$

$(A - r^2 I)\vec{c}e^{rt} = 0$

$A - r^2 I = 0$

$|A - r^2 I| = 0$ by def of determinant & char. eq.

[1] **Reminder**: Linear algebra writes eigenpair equation $A\vec{x} = \lambda\vec{x}$ equivalently as $A\vec{x} = \lambda I\vec{x}$ and then converts it to the homogeneous system of linear algebraic equations $(A - \lambda I)\vec{x} = \vec{0}$. The proof you write should apply without edits to $n \times n$ real matrices A.

(6b) [40%] Substitute $\vec{\mathbf{u}}(t) = \vec{\mathbf{d}}\cos(5t)$ into $\frac{d^2}{dt^2}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ to determine vector $\vec{\mathbf{d}}$ in terms of eigenpairs of A. Repeat for $\vec{\mathbf{u}}(t) = \vec{\mathbf{d}}\sin(5t)$ and report the answer.

A

$\vec{u}(t) = \vec{d}\cos 5t$

$A = \begin{pmatrix} -13 & -6 \\ 6 & -28 \end{pmatrix}$

$\frac{d}{dt}\vec{u}(t) = -5\vec{d}\sin 5t$

$\left(-25, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$

$\frac{d^2}{dt} = -25\vec{d}\cos 5t = A\vec{d}\cos 5t$

$\left(-16, \begin{pmatrix} 3 \\ 1 \end{pmatrix}\right)$

$-25\vec{d} = A\vec{d}$

$\boxed{\vec{d} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$ by def of eigenval / eigenpair

---

$\vec{u}(t) = \vec{d}\sin 5t$

$\vec{u}' = 5\vec{d}\cos 5t$

$u'' = -25\vec{d}\sin 5t$

$-25\vec{d}\sin 5t = A\vec{d}\cancel{\sin 5t}$ by subst.

$-25\vec{d} = A\vec{d}$

$\boxed{\vec{d} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \text{ by def. of eigen val / eigen pair}}$

(6c) [30%] The Euler solution atoms $\cos(5t), \sin(5t), \cos(4t), \sin(4t)$ are linearly independent on $(-\infty, \infty)$. Substitute $\vec{\mathbf{u}}(t) = \vec{\mathbf{d}}_1 \cos(5t) + \vec{\mathbf{d}}_2 \sin(5t) + \vec{\mathbf{d}}_3 \cos(4t) + \vec{\mathbf{d}}_4 \sin(4t)$ into $\frac{d^2}{dt^2}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ and use independence (vector coefficients of atoms match) to determine the vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3, \vec{\mathbf{d}}_4$ in terms of eigenpairs of A.

A

$\left(\vec{u} = d_1\cos 5t + d_2 \sin 5t + d_3 \cos 4t + d_4 \sin(4t)\right)$

$u'' = \left(-25 d_1 \cos 5t - 25 d_2 \sin 5t - 16 d_3 \cos 4t - 16 d_4 \sin 4t = \right) A\vec{u}$

$= A$

$-25\vec{d}_1 = A\vec{d}_1$

$-25\vec{d}_2 = A\vec{d}_2$

$-16\vec{d}_3 = A\vec{d}_3$

$-16\vec{d}_4 = A\vec{d}_4$

By independence of atoms. (cancellation of like terms like prev. parts)

$d_1 = \begin{pmatrix}1\\2\end{pmatrix}$ $d_2 = \begin{pmatrix}1\\2\end{pmatrix}$ $d_3 = \begin{pmatrix}2\\1\end{pmatrix}$ $d_4 = \begin{pmatrix}2\\1\end{pmatrix}$ ← multiply each by c_1, c_2, c_3, c_4. Eigenvectors are not unique, but eigenspaces are 1-dimensional.

by def of eigenpairs given.