# Differential Equations 2280

**Midterm Exam 2 with Solutions**

**Exam Date: 5 April 2019 at 7:30am**

**Instructions**: This in-class exam is 80 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

## Laplace Theory: Chapter 7

**Laplace Theory Background**. Expected without notes or books is the 4-entry Forward Laplace Table and these rules: Linearity, Parts, Shift, $s$-differentiation, Lerch's Theorem, Final Value Theorem, Existence theory.

1. **(Laplace Theory: Differential Equations)**
   **Solve differential equations (1a), (1b) by Laplace's method.**

   **(1a)** [40%] Third order $x''' + x' = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$.

   **(1b)** [60%] Dynamical system $x' = x + y$, $y' = 2x + 6$, $x(0) = 0$, $y(0) = 0$.

2. **(Laplace Theory: Backward Table)**
   **Solve for $f(t)$ in (2a), (2b), (2c), (2d).**

   **Assumptions**. Below, $f(t)$ is of piecewise continuous of exponential order. Expression $u(t)$ denotes the unit step function.

   **Credit**. Document all steps, e.g., if you cancel $\mathcal{L}$ then cite Lerch's Theorem. The answer is 25%. The documented steps are 75%. Partial fraction coefficients are expected to be evaluated last.

   **(2a)** [20%] $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+25)(s^2+4)}$

   **(2b)** [20%] $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2+4s+29}$.

   **(2c)** [30%] $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+2s)(s^2-3s)}$

   **(2d)** [30%] $\dfrac{d}{ds}\mathcal{L}(f(t))\Big|_{s\to(s-3)} = \dfrac{d}{ds}\mathcal{L}(u(t-\pi)e^t\cos 25t)$.

3. **(Laplace Theory: Forward Table)**
   **Compute the Laplace transform $\mathcal{L}(f(t))$.**

   **(3a)** [20%] $f(t) = (-t)e^{2t}\sin(3t)$.

   **(3b)** [30%] $f(t)) = e^{-\pi t}g(t)$ and $g(t) = \dfrac{e^{2t}-e^{-2t}}{t}$.

   **(3c)** [20%] $\dfrac{d}{ds}\mathcal{L}(f(t)) = \dfrac{1}{s^{2019}} + \dfrac{1}{s^2+1}$.

   **(3d)** [30%] Define $f(t) = e^{-2t}g(t))$ where $g(t) = \dfrac{e^{2t}-e^{-2t}}{t}$.

# Systems of Differential Equations: Chapters 4 and 5

## Background.

The **Eigenanalysis Method** for a real $3 \times 3$ matrix $A$ assumes eigenpairs $(\lambda_1, \vec{\mathbf{v}}_1)$, $(\lambda_2, \vec{\mathbf{v}}_2)$, $(\lambda_3, \vec{\mathbf{v}}_3)$. It says that the $3 \times 3$ system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ has general solution $\vec{\mathbf{x}}(t) = c_1\vec{\mathbf{v}}_1 e^{\lambda_1 t} + c_2\vec{\mathbf{v}}_2 e^{\lambda_2 t} + c_3\vec{\mathbf{v}}_3 e^{\lambda_3 t}$.

The **Cayley-Hamilton-Ziebur method** is based upon this result:

> Let $A$ be an $n \times n$ real matrix. The components of solution $\vec{\mathbf{u}}$ of $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ are linear combinations of Euler solution atoms obtained from the roots of the characteristic equation $|A - \lambda I| = 0$. Alternatively, $\vec{\mathbf{u}}(t)$ is a vector linear combination of the Euler solution atoms: $\vec{\mathbf{u}}(t) = \sum_{k=1}^{n}(atom_k)\vec{\mathbf{d}}_k$.

A **Fundamental Matrix** is an $n \times n$ matrix $\Phi(t)$ with columns consisting of independent solutions of $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$, where $A$ is an $n \times n$ real matrix. The general solution of $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t)$ is $\vec{\mathbf{x}}(t) = \Phi(t)\vec{\mathbf{c}}$, where $\vec{\mathbf{c}}$ is a column vector of arbitrary constants $c_1, \ldots, c_n$. An alternate and widely used definition of fundamental matrix is $\Phi'(t) = A\Phi(t)$, with $|\Phi(0)| \neq 0$ required to establish independence of the columns of $\Phi$.

The **Exponential Matrix**, denoted $e^{At}$, is the unique fundamental matrix $\Psi(t)$ such that $\Psi(0) = I$. Matrix $A$ is an $n \times n$ real matrix. It is known that $e^{At} = \Phi(t)\Phi(0)^{-1}$ for *any* fundamental matrix $\Phi(t)$. Consequently, $\frac{d}{dt}\left(e^{At}\right) = A\,e^{At}$ and $e^{At}\Big|_{t=0} = I$.

**4. (Systems: Eigenanalysis Method)**
Complete parts **((4a), (4b), (4c)**.

**(4a)** [40%] Let $A = \begin{pmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{pmatrix}$. Display the linear algebra details for computing the three eigenpairs of $A$.

**(4b)** [30%] Matrix $A = \begin{pmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{pmatrix}$ has eigenpairs $\left(2, \begin{pmatrix} 1 \\ -5 \\ -3 \\ 3 \end{pmatrix}\right)$, $\left(3, \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$, $\left(4, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)$, $\left(5, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}\right)$. Display the general solution of $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$.

**(4c)** [40%] Let $\vec{\mathbf{u}}(t) = \vec{\mathbf{c}}\,e^{rt}$, the vector Euler substitution. Assume real $n \times n$ matrix $A$ has a real eigenpair $(\lambda, \vec{\mathbf{v}})$. Prove that the Euler substitution with $r = \lambda$ and $\vec{\mathbf{c}} = \vec{\mathbf{v}}$ applied to $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ finds a nonzero solution of $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$.

**5. (Systems: First Order Cayley-Hamilton-Ziebur)**

**(5a)** [30%] The eigenvalues are 2, 6 for the matrix $A = \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$.

Display the general solution of $\frac{d}{dt}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5).

**(5b)** [40%] The $3 \times 3$ triangular matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 5 & 1 \\ 0 & 0 & -2 \end{pmatrix}$ represents a linear cascade, such as found in brine tank models. Apply the **linear integrating factor method** and related shortcuts (Ch 1 in the textbook) to find the components $x_1, x_2, x_3$ of the vector general solution $\vec{\mathbf{x}}(t)$ of $\frac{d}{dt}\vec{\mathbf{x}}(t) = A\vec{\mathbf{x}}(t)$.

**(5c)** [30%] The Cayley-Hamilton-Ziebur shortcut applies to the system

$$x' = 3x + 2y, \quad y' = -2x + 3y,$$

which has complex eigenvalues $\lambda = 3 \pm 2i$. Find a fundamental matrix $\Phi(t)$ for this system, documenting all details of the computation.

**6. (Systems: Second Order Cayley-Hamilton-Ziebur)**

Assume below that real $2 \times 2$ matrix $A = \begin{pmatrix} -13 & -6 \\ 6 & -28 \end{pmatrix}$ has eigenpairs $\left(-25, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$, $\left(-16, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right)$. Textbook theorems applied to $\frac{d^2}{dt^2}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ report general solution

$$\vec{\mathbf{u}}(t) = (c_1 \cos(5t) + c_2 \sin(5t) \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (c_3 \cos(4t) + c_4 \sin(4t)) \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

**(6a)** [30%] Derive the characteristic equation $|A - r^2 I| = 0$ for the second order equation $\frac{d^2}{dt^2}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ from Euler's vector substitution $\vec{\mathbf{u}}(t) = \vec{\mathbf{c}}\, e^{rt}$. A proof is expected with details.[1]

**(6b)** [40%] Substitute $\vec{\mathbf{u}}(t) = \vec{\mathbf{d}} \cos(5t)$ into $\frac{d^2}{dt^2}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ to determine vector $\vec{\mathbf{d}}$ in terms of eigenpairs of $A$. Repeat for $\vec{\mathbf{u}}(t) = \vec{\mathbf{d}} \sin(5t)$ and report the answer.

**(6c)** [30%] The Euler solution atoms $\cos(5t), \sin(5t), \cos(4t), \sin(4t)$ are linearly independent on $(-\infty, \infty)$. Substitute $\vec{\mathbf{u}}(t) = \vec{\mathbf{d}}_1 \cos(5t) + \vec{\mathbf{d}}_2 \sin(5t) + \vec{\mathbf{d}}_3 \cos(4t) + \vec{\mathbf{d}}_4 \sin(4t)$ into $\frac{d^2}{dt^2}\vec{\mathbf{u}}(t) = A\vec{\mathbf{u}}(t)$ and use independence (vector coefficients of atoms match) to determine the vectors $\vec{\mathbf{d}}_1, \vec{\mathbf{d}}_2, \vec{\mathbf{d}}_3, \vec{\mathbf{d}}_4$ in terms of eigenpairs of $A$.

[1] **Reminder**: Linear algebra writes eigenpair equation $A\vec{\mathbf{x}} = \lambda\vec{\mathbf{x}}$ equivalently as $A\vec{\mathbf{x}} = \lambda I \vec{\mathbf{x}}$ and then converts it to the homogeneous system of linear algebraic equations $(A - \lambda I)\vec{\mathbf{x}} = \vec{\mathbf{0}}$. The proof you write should apply without edits to $n \times n$ real matrices $A$.