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Differential Equations 2280

Midterm Exam 1

Exam Date: Friday, 22 February 2019 at 7:45am in LCB 215

Instructions: This in-class exam is designed for 80 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

A (a) [40%] Solve $y' = \dfrac{3x^4}{1+x^2}$.

A (b) [60%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}((1+t)v(t)) = 3t^2$, $v(0) = 0$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 0$.

(a) $\int \frac{dy}{dx}dx = \int \frac{3x^4}{1+x^2}\,dx$

$$x^2+1\,\overline{)\,3x^4\,} \quad \text{quotient } 3x^2 - 3 + \frac{3}{1+x^2}$$
$$-(3x^4+3x^2)$$
$$-3x^2$$
$$-(-3x^2-3)$$
$$\boxed{3}$$

$y + c_1 = \int 3x^2 - 3\,dx + \int \frac{+3}{1+x^2}\,dx$

$$\boxed{y = x^3 - 3x + 3\tan^{-1}(x) + c}$$

(b) $\int \frac{d}{dt}[(1+t)v(t)]dt = \int 3t^2\,dt$

$(1+t)v = t^3 + c$ $\quad v(0)=0$ $\quad \boxed{c=0}$

$v = \frac{t^3}{1+t} + \frac{c}{1+c}$

$v = \frac{t^3}{1+t}$

$\int x'dt = \int \frac{t^3}{1+t}\,dt$

$$t+1\,\overline{)\,t^3\,} \quad \text{quotient } t^2 - t + 1 - \frac{1}{t+1}$$
$$-(t^3+t^2)$$
$$-t^2$$
$$-(t^2-t)$$
$$t$$
$$-(t+1)$$
$$-1$$

$x + c_2 = \int t^2 - t + 1 - \frac{1}{t+1}\,dt$

$x = \frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1| + c$

$x(0)=0$ $\boxed{c=0}$ $\Rightarrow c=0$

$0 = 0 - 0 + 0 - 0 + c$

$$\boxed{x = \frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|}$$

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2. (Solve a Separable Equation)

Given $(5y+15)y' = \left(\frac{x}{1+x} + \tan(x)\right)(y^2 - 3y + 4)$.

y^2-3y+4 was supposed to be y^2-3y-4.
A single character typo.

A

Find a non-equilibrium solution in implicit form.
To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

$$(5y+15)y' = \left(\frac{x}{1+x} + \tan x\right)(y^2-3y+4)$$

$$\Rightarrow y' = \left(\frac{x}{1+x} + \tan x\right)\frac{y^2-3y+4}{5y+15}$$

$$\Rightarrow F(x) = \frac{x}{1+x} + \tan x, \quad G(y) = \frac{y^2-3y+4}{5y+15}$$

$$\Rightarrow \int \frac{dy}{G(y)} = \int F(x)\,dx$$

$$\Rightarrow \int F(x)\,dx = \int \frac{x}{1+x}\,dx + \int \frac{\sin x}{\cos x}\,dx$$

$$= \int \frac{u-1}{u}\,du + \int \frac{-1}{v}\,dv$$

$$= u - \ln|u| - \ln|v| + C$$

$$= x - \ln|x+1| - \ln|\cos x| + C$$

$$\Rightarrow \int \frac{dy}{G(y)} = \int \frac{5y+15}{y^2-3y+4}\,dy$$

$$= \int \frac{5y}{y^2-3y+4}\,dy + \int \frac{15}{y^2-3y+4}\,dy$$

$$= \text{some function } M(y)$$

$$\Rightarrow M(y) = x - \ln|x+1| - \ln|\cos x| + C$$

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3. (Linear Equations)

A (a) [60%] Solve the linear velocity model $v'(t) = -32 - \frac{1}{t+1}v(t)$, $v(0) = 0$. Show all integrating factor steps.

A (b) [20%] Solve $(x+2)\frac{dy}{dx} - x\,y = 0$ using the homogeneous linear equation shortcut.

A (c) [20%] Solve $3\frac{dy}{dx} + 4y = \frac{11}{5}$ using the superposition principle $y = y_h + y_p$ shortcut. Expected are answers for y_h and y_p.

a) We have $V' = -32 - \frac{1}{t+1}V$

Rearrange to std. form $\Rightarrow V' + \frac{1}{t+1}V = -32$

Formula for integrating factor $\Rightarrow P(t) = e^{\int \frac{1}{t+1}dt} = e^{\ln|t+1|} = t+1$

Multiply both sides by $P(t) \Rightarrow (t+1)(V' + \frac{1}{t+1}V) = -32(t+1)$

Left side reduces by int. fact. $\Rightarrow \frac{d}{dt}(V(t+1)) = -32(t+1)$

Integrate both sides $\Rightarrow \int \frac{d}{dt}(V(t+1))dt = \int -32(t+1)dt$

Evaluate integrals $\Rightarrow V(t+1) = -16(t+1)^2 + C$

Divide both sides by $t+1$ $\Rightarrow V = -16t - 16 + \frac{C}{t+1}$

Plug in init. cond. $V(0) = 0$ $\Rightarrow 0 = -16 + \frac{C}{0+1} \Rightarrow C = 16$

Plug in $C = 16$ $\Rightarrow V = -16t - 16 + \frac{16}{t+1}$

b) We have $(x+2)\frac{dy}{dx} - xy = 0$

Rearrange to std. form $\Rightarrow \frac{dy}{dx} - \frac{x}{x+2}y = 0$

The integrating factor is $e^{\int \frac{x}{x+2}dx} = e^{x - 2\ln|x+2|} = \frac{e^x}{(x+2)^2}$

By the homogenous linear equation shortcut, we know

$$y = \frac{C}{P(x)} = \frac{C}{\left(\frac{e^x}{(x+2)^2}\right)} = \frac{C(x+2)^2}{e^x}$$

c) We have $3\frac{dy}{dx} + 4y = \frac{11}{5}$

std. form $\Rightarrow \frac{dy}{dx} + \frac{4}{3}y = \frac{11}{15}$ $\qquad \Rightarrow 3 \times 0 + 4y_p = \frac{11}{5}$

Int. fact. form $\Rightarrow P(x) = e^{\int \frac{4}{3}dx} = e^{4/3 x}$ $\qquad \Rightarrow y_p = \frac{11}{20}$

Form. for $y_h \Rightarrow y_h = \frac{C}{P(x)} = \frac{C}{e^{4/3 x}}$

$$\Rightarrow y = y_h + y_p = \frac{11}{20} + \frac{C}{e^{\frac{4}{3}x}}$$

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4. (Stability)

Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase portrait with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.

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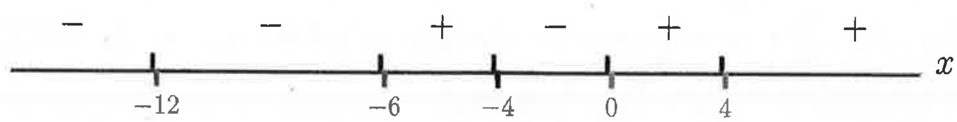

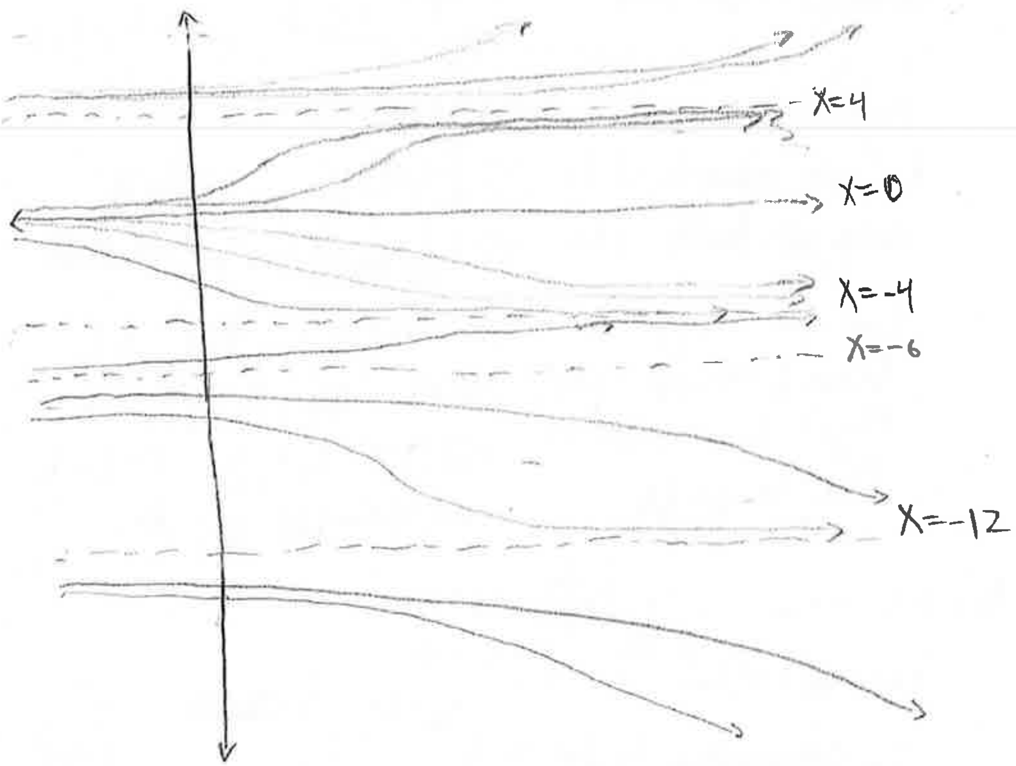

@ x=4 node, unstable
@ x=0 spout, unstable
@ x=-4 funnel, stable
@ x=-6 spout, unstable
@ x=-12 node, unstable

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5. (Chapter 3: Linear nth Order DE)
Using Euler's theorem on Euler solution atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c).

A (a) [40%] Find a constant coefficient differential equation $ay'' + by' + cy = 0$ which has a particular solution $10e^{-2x} + xe^{-2x}$.

A (b) [30%] Given characteristic equation $r(r+2)(r^3 - 4r)(r^2 + 2r + 65) = 0$, solve the differential equation.

A (c) [30%] Given $mx''(t) + cx'(t) + kx(t) = 0$, which represents an unforced damped spring-mass system. Assume $m = 4$, $c = 4$, $k = 101$. Classify the equation as over-damped, critically damped or under-damped. Illustrate in a spring-mass-dashpot drawing the assignment of physical constants m, c, k and the initial conditions $x(0) = -1$, $x'(0) = 0$.

a) We see that e^{-2x} and xe^{-2x} are solution atoms to the diff. eq. Therefore, the characteristic equation has a root at -2 of multiplicity 2. Since the diff eq is 2nd order, the characteristic equation must be

$$(r+2)^2 = 0 \Rightarrow r^2 + 4r + 4 = 0$$

So the diff eq. is $y'' + 4y' + 4y = 0$,

b) $r(r+2)(r^3-4r)(r^2+2r+65) = 0$

$\Rightarrow r^2(r+2)^2(r-2)(r^2+2r+65) = 0$

$\Rightarrow r = 0$ (mult. 2), -2 (mult. 2), 2 (mult. 1), $-1 \pm 8i$

The solution atoms are therefore $1, x, e^{-2x}, xe^{-2x}, e^{2x}, e^{-x}\cos 8x, e^{-x}\sin 8x$

So the solution to the diff eq. is $y = C_1 + C_2 x + C_3 e^{-2x} + C_4 xe^{-2x} + C_5 e^{2x} + C_6 e^{-x}\cos 8x + C_7 e^{-x}\sin 8x$

c) We have $4x'' + 4x' + 101x = 0$

$\Rightarrow 4r^2 + 4r + 101 = 0$

$\Rightarrow r = \frac{-4 \pm \sqrt{16 - 4\times 101 \times 4}}{8}$

$\Rightarrow r = \frac{-4 \pm \sqrt{-1600}}{8} = -\frac{1}{2} \pm 5i$

$\Rightarrow$ atoms $= e^{-\frac{1}{2}x}\cos 5x, e^{-\frac{1}{2}x}\sin 5x$

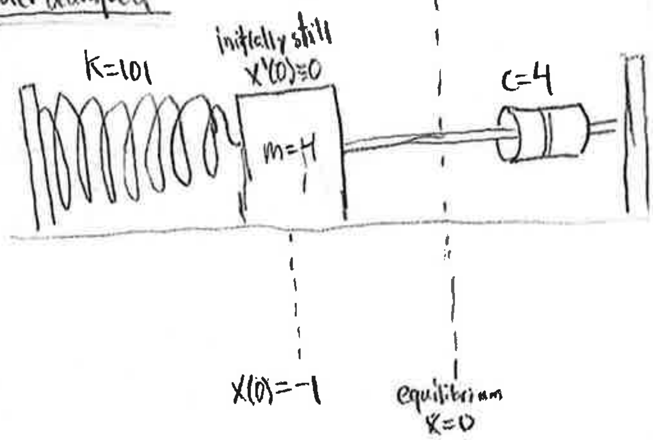
$\Rightarrow x = e^{-\frac{1}{2}x}(A\cos 5x + B\sin 5x)$

$\Rightarrow -1 = e^{-\frac{1}{2}\cdot 0}(A\cos 0 + B\sin 0) \Rightarrow A = -1$

$\Rightarrow x' =$

don't need to solve

Because the determinant is negative, the system is underdamped



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6. (Chapter 3: Linear nth Order DE)

Determine the corrected trial solution for y_p according to the method of undetermined coefficients for equation

A

$$y^{(4)} + y^{(2)} = x + 2e^x + \cos(x).$$

Do not evaluate the undetermined coefficients! The trial solution should be the one with fewest Euler solution atoms.

The characteristic equation for the homogenous equation is

$$r^4 + r^2 = 0 \Rightarrow r^2(r^2+1) = 0$$
$$\Rightarrow r = 0, 0, \pm i$$

The solution atoms are $1, x, \cos x, \sin x$, so the complementary function is $y_h = C_1 + C_2 x + C_3 \cos x + C_4 \sin x$.

All derivatives of the RHS of the nonhomogenous equation gives us following atoms:

$$y_1 = C_1 + C_2 x$$
$$y_2 = C_3 e^x$$
$$y_3 = C_4 \cos x$$
$$y_4 = C_5 \sin x$$

By the method of undet. coeff., we multiply each by x^s until they are linearly independent of the atoms for the homogenous equation

$$\Rightarrow y_1 = C_1 x^2 + C_2 x^3$$
$$y_2 = C_3 e^x$$
$$y_3 = C_4 x \cos x$$
$$y_4 = C_5 x \sin x$$

$$\Rightarrow y_p = y_1 + y_2 + y_3 + y_4 = C_1 x^2 + C_2 x^3 + C_3 e^x + C_4 x \cos x + C_5 x \sin x$$