11.9 Numerical Methods for Systems

An initial value problem for a system of two differential equations is given by the equations

$$
\begin{aligned}
x'(t) &= f(t, x(t), y(t)), \\
y'(t) &= g(t, x(t), y(t)), \\
x(t_0) &= x_0, \\
y(t_0) &= y_0.
\end{aligned} \tag{1}
$$

A **numerical method** for (1) is an algorithm that computes an approximate dot table with first line t_0, x_0, y_0. Generally, the dot table has equally spaced t-values, two consecutive t-values differing by a constant value $h \neq 0$, called the **step size**. To illustrate, if $t_0 = 2$, $x_0 = 5$, $y_0 = 100$, then a typical dot table for step size $h = 0.1$ might look like

t	x	y
2.0	5.00	100.00
2.1	5.57	103.07
2.2	5.62	104.10
2.3	5.77	102.15
2.4	5.82	101.88
2.5	5.96	100.55

Graphics. The dot table represents the data needed to plot a solution curve to system (1) in three dimensions (t, x, y) or in two dimensions, using a tx-scene or a ty-scene. In all cases, the plot is a simple connect-the-dots graphic.

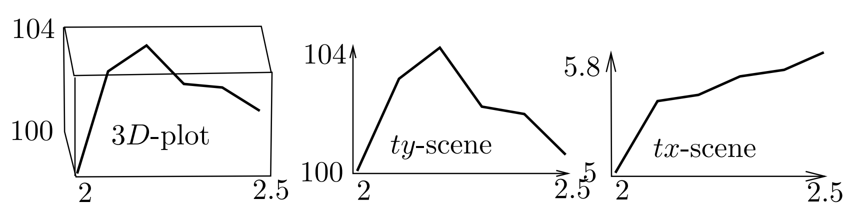

Figure 23. Dot table plots.
The three dimensional plot is a space curve made directly from the dot table. The tx-scene and the ty-scene are made from the same dot table using corresponding data columns.

Myopic Algorithms. All of the popular algorithms for generating a numerical dot table for system (1) are **near-sighted**, because they predict the next line in the dot table from the current dot table line, ignoring effects and errors for all other preceding dot table lines. Among such algorithms are **Euler's method**, **Heun's method** and the **RK4 method**.

Numerical Algorithms: Planar Case

Stated here without proof are three numerical algorithms for solving two-dimensional initial value problems (1). Justification of the formulas is obtained from the vector relations in the next subsection.

Notation. Let t_0, x_0, y_0 denote the entries of the dot table on a particular line. Let h be the increment for the dot table and let t_0+h, x, y stand for the dot table entries on the next line.

Planar Euler Method.

$$\begin{aligned} x &= x_0 + hf(t_0, x_0, y_0), \\ y &= y_0 + hg(t_0, x_0, y_0). \end{aligned}$$

Planar Heun Method.

$$\begin{aligned} x_1 &= x_0 + hf(t_0, x_0, y_0), \\ y_1 &= y_0 + hg(t_0, x_0, y_0), \\ x &= x_0 + h(f(t_0, x_0, y_0) + f(t_0+h, x_1, y_1))/2 \\ y &= y_0 + h(g(t_0, x_0, y_0) + g(t_0+h, x_1, y_1))/2. \end{aligned}$$

Planar RK4 Method.

$$\begin{aligned} k_1 &= hf(t_0, x_0, y_0), \\ m_1 &= hg(t_0, x_0, y_0), \\ k_2 &= hf(t_0+h/2, x_0+k_1/2, y_0+m_1/2), \\ m_2 &= hg(t_0+h/2, x_0+k_1/2, y_0+m_1/2), \\ k_3 &= hf(t_0+h/2, x_0+k_2/2, y_0+m_2/2), \\ m_3 &= hg(t_0+h/2, x_0+k_2/2, y_0+m_2/2), \\ k_4 &= hf(t_0+h, x_0+k_3, y_0+m_3), \\ m_4 &= hg(t_0+h, x_0+k_3, y_0+m_3), \\ x &= x_0 + \frac{1}{6}\left(k_1+2k_2+2k_3+k_4\right), \\ y &= y_0 + \frac{1}{6}\left(m_1+2m_2+2m_3+m_4\right). \end{aligned}$$

Numerical Algorithms: General Case

Consider a vector initial value problem

$$\vec{\mathbf{u}}'(t) = \vec{\mathbf{F}}(t, \vec{\mathbf{u}}(t)), \quad \vec{\mathbf{u}}(t_0) = \vec{\mathbf{u}}_0.$$

Described here are the vector formulas for Euler, Heun and RK4 methods. These myopic algorithms predict the next table dot t_0+h, $\vec{\mathbf{u}}$ from the current dot t_0, $\vec{\mathbf{u}}_0$. The number of scalar values in a table dot is $1+n$, where n is the dimension of the vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{F}}$.

Vector Euler Method.

$$\vec{\mathbf{u}} = \vec{\mathbf{u}}_0 + h\vec{\mathbf{F}}(t_0, \vec{\mathbf{u}}_0)$$

Vector Heun Method.

$$\vec{\mathbf{w}} = \vec{\mathbf{u}}_0 + h\vec{\mathbf{F}}(t_0, \vec{\mathbf{u}}_0), \quad \vec{\mathbf{u}} = \vec{\mathbf{u}}_0 + \frac{h}{2}\left(\vec{\mathbf{F}}(t_0, \vec{\mathbf{u}}_0) + \vec{\mathbf{F}}(t_0 + h, \vec{\mathbf{w}})\right)$$

Vector RK4 Method.

$$\begin{aligned}
\vec{\mathbf{k}}_1 &= h\vec{\mathbf{F}}(t_0, \vec{\mathbf{u}}_0), \\
\vec{\mathbf{k}}_1 &= h\vec{\mathbf{F}}(t_0 + h/2, \vec{\mathbf{u}}_0 + \vec{\mathbf{k}}_1/2), \\
\vec{\mathbf{k}}_1 &= h\vec{\mathbf{F}}(t_0 + h/2, \vec{\mathbf{u}}_0 + \vec{\mathbf{k}}_2/2), \\
\vec{\mathbf{k}}_1 &= h\vec{\mathbf{F}}(t_0 + h, \vec{\mathbf{u}}_0 + \vec{\mathbf{k}}_3), \\
\vec{\mathbf{u}} &= \vec{\mathbf{u}}_0 + \frac{1}{6}\left(\vec{\mathbf{k}}_1 + 2\vec{\mathbf{k}}_2 + 2\vec{\mathbf{k}}_3 + \vec{\mathbf{k}}_4\right).
\end{aligned}$$