11.7 Nonhomogeneous Linear Systems

Variation of Parameters

The **method of variation of parameters** is a general method for solving a linear nonhomogeneous system

$$\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t).$$

Historically, it was a trial solution method, whereby the nonhomogeneous system is solved using a trial solution of the form

$$\vec{\mathbf{x}}(t) = e^{At}\,\vec{\mathbf{x}}_0(t).$$

In this formula, $\vec{\mathbf{x}}_0(t)$ is a vector function to be determined. The method is imagined to originate by varying $\vec{\mathbf{x}}_0$ in the general solution $\vec{\mathbf{x}}(t) = e^{At}\,\vec{\mathbf{x}}_0$ of the linear homogenous system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$. Hence was coined the names *variation of parameters* and *variation of constants*.

Modern use of variation of parameters is through a formula, memorized for routine use.

Theorem 28 (Variation of Parameters for Systems)
Let A be a constant $n \times n$ matrix and $\vec{\mathbf{F}}(t)$ a continuous function near $t = t_0$. The unique solution $\vec{\mathbf{x}}(t)$ of the matrix initial value problem

$$\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t), \quad \vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0,$$

is given by the **variation of parameters formula**

$$\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{x}}_0 + e^{At}\int_{t_0}^{t} e^{-rA}\vec{\mathbf{F}}(r)dr. \tag{1}$$

Proof of (1). Define

$$\vec{\mathbf{u}}(t) = \vec{\mathbf{x}}_0 + \int_{t_0}^{t} e^{-rA}\vec{\mathbf{F}}(r)dr.$$

To show (1) holds, we must verify $\vec{\mathbf{x}}(t) = e^{At}\vec{\mathbf{u}}(t)$. First, the function $\vec{\mathbf{u}}(t)$ is differentiable with continuous derivative $e^{-tA}\vec{\mathbf{F}}(t)$, by the fundamental theorem of calculus applied to each of its components. The product rule of calculus applies to give

$$\begin{aligned}\vec{\mathbf{x}}'(t) &= \left(e^{At}\right)'\vec{\mathbf{u}}(t) + e^{At}\vec{\mathbf{u}}'(t)\\ &= Ae^{At}\vec{\mathbf{u}}(t) + e^{At}e^{-At}\vec{\mathbf{F}}(t)\\ &= A\vec{\mathbf{x}}(t) + \vec{\mathbf{F}}(t).\end{aligned}$$

Therefore, $\vec{\mathbf{x}}(t)$ satisfies the differential equation $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$. Because $\vec{\mathbf{u}}(t_0) = \vec{\mathbf{x}}_0$, then $\vec{\mathbf{x}}(t_0) = \vec{\mathbf{x}}_0$, which shows the initial condition is also satisfied. The proof is complete.

Undetermined Coefficients

The trial solution method known as the method of undetermined coefficients can be applied to vector-matrix systems $\vec{\mathbf{x}}' = A\vec{\mathbf{x}} + \vec{\mathbf{F}}(t)$ when the components of $\vec{\mathbf{F}}$ are sums of terms of the form

$$(\text{polynomial in } t)e^{at}(\cos(bt) \text{ or } \sin(bt)).$$

Such terms are known as **Euler solution atoms**. It is usually efficient to write $\vec{\mathbf{F}}$ in terms of the columns $\vec{\mathbf{e}}_1, \ldots, \vec{\mathbf{e}}_n$ of the $n \times n$ identity matrix I, as the combination

$$\vec{\mathbf{F}}(t) = \sum_{j=1}^{n} F_j(t)\vec{\mathbf{e}}_j.$$

Then

$$\vec{\mathbf{x}}(t) = \sum_{j=1}^{n} \vec{\mathbf{x}}_j(t),$$

where $\vec{\mathbf{x}}_j(t)$ is a particular solution of the simpler equation

$$\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t) + f(t)\vec{\mathbf{c}}, \quad f = F_j, \quad \vec{\mathbf{c}} = \vec{\mathbf{e}}_j.$$

An initial trial solution $\vec{\mathbf{x}}(t)$ for $\vec{\mathbf{x}}'(t) = A\vec{\mathbf{x}}(t) + f(t)\vec{\mathbf{c}}$ can be determined from the following **initial trial solution rule**:

> Let $f(t)$ be a sum of Euler solution atoms. Identify independent functions whose linear combinations give all derivatives of $f(t)$. The initial trial solution is a linear combination of these functions with undetermined vector coefficients $\{\vec{\mathbf{c}}_j\}$.

In the well-known scalar case, the trial solution must be modified if its terms contain any portion of the general solution to the homogeneous equation. In the vector case, if $f(t)$ is a polynomial, then the *correction rule* for the initial trial solution is avoided by assuming the matrix A is invertible. This assumption means that $r = 0$ is not a root of $\det(A - rI) = 0$, which prevents the homogenous solution from having any polynomial terms.

The initial vector trial solution is substituted into the differential equation to find the undetermined coefficients $\{\vec{\mathbf{c}}_j\}$, hence finding a particular solution.

Theorem 29 (Polynomial solutions)
Let $f(t) = \sum_{j=0}^{k} p_j \frac{t^j}{j!}$ be a polynomial of degree k. Assume A is an $n \times n$ constant invertible matrix. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + f(t)\vec{\mathbf{c}}$ has a polynomial solution $\vec{\mathbf{u}}(t) = \sum_{j=0}^{k} \vec{\mathbf{c}}_j \frac{t^j}{j!}$ of degree k with vector coefficients $\{\vec{\mathbf{c}}_j\}$ given by the relations

$$\vec{\mathbf{c}}_j = -\sum_{i=j}^{k} p_i A^{j-i-1}\vec{\mathbf{c}}, \quad 0 \le j \le k.$$

Theorem 30 (Polynomial $\times$ exponential solutions)
Let $g(t) = \sum_{j=0}^{k} p_j \frac{t^j}{j!}$ be a polynomial of degree k. Assume A is an $n \times n$ constant matrix and $B = A - aI$ is invertible. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + e^{at}g(t)\vec{\mathbf{c}}$ has a polynomial-exponential solution $\vec{\mathbf{u}}(t) = e^{at}\sum_{j=0}^{k} \vec{\mathbf{c}}_j \frac{t^j}{j!}$ with vector coefficients $\{\vec{\mathbf{c}}_j\}$ given by the relations

$$\vec{\mathbf{c}}_j = -\sum_{i=j}^{k} p_i B^{j-i-1}\vec{\mathbf{c}}, \quad 0 \le j \le k.$$

Proof of Theorem 29. Substitute $\vec{\mathbf{u}}(t) = \sum_{j=0}^{k} \vec{\mathbf{c}}_j \frac{t^j}{j!}$ into the differential equation, then

$$\sum_{j=0}^{k-1} \vec{\mathbf{c}}_{j+1} \frac{t^j}{j!} = A\sum_{j=0}^{k} \vec{\mathbf{c}}_j \frac{t^j}{j!} + \sum_{j=0}^{k} p_j \frac{t^j}{j!}\vec{\mathbf{c}}.$$

Then terms on the right for $j = k$ must add to zero and the others match the left side coefficients of $t^j/j!$, giving the relations

$$A\vec{\mathbf{c}}_k + p_k\vec{\mathbf{c}} = \vec{\mathbf{0}}, \quad \vec{\mathbf{c}}_{j+1} = A\vec{\mathbf{c}}_j + p_j\vec{\mathbf{c}}.$$

Solving these relations recursively gives the formulas

$$\begin{array}{lcl} \vec{\mathbf{c}}_k & = & -p_k A^{-1}\vec{\mathbf{c}}, \\ \vec{\mathbf{c}}_{k-1} & = & -\left(p_{k-1}A^{-1} + p_k A^{-2}\right)\vec{\mathbf{c}}, \\ & \vdots & \\ \vec{\mathbf{c}}_0 & = & -\left(p_0 A^{-1} + \cdots + p_k A^{-k-1}\right)\vec{\mathbf{c}}. \end{array}$$

The relations above can be summarized by the formula

$$\vec{\mathbf{c}}_j = -\sum_{i=j}^{k} p_i A^{j-i-1}\vec{\mathbf{c}}, \quad 0 \le j \le k.$$

The calculation shows that if $\vec{\mathbf{u}}(t) = \sum_{j=0}^{k} \vec{\mathbf{c}}_j \frac{t^j}{j!}$ and $\vec{\mathbf{c}}_j$ is given by the last formula, then $\vec{\mathbf{u}}(t)$ substituted into the differential equation gives matching LHS and RHS. The proof is complete.

Proof of Theorem 30. Let $\vec{\mathbf{u}}(t) = e^{at}\vec{\mathbf{v}}(t)$. Then $\vec{\mathbf{u}}' = A\vec{\mathbf{u}} + e^{at}g(t)\vec{\mathbf{c}}$ implies $\vec{\mathbf{v}}' = (A - aI)\vec{\mathbf{v}} + g(t)\vec{\mathbf{c}}$. Apply Theorem 29 to $\vec{\mathbf{v}}' = B\vec{\mathbf{v}} + g(t)\vec{\mathbf{c}}$. The proof is complete.