Final Exam Solutions S2019 2270-4

1. (Chapter 1: 100 points) Solve Linear Algebraic Equations.

Solve the system $A\vec{u} = \vec{b}$ defined by

$$\begin{cases} 2x_1 + x_2 + 8x_3 + x_4 + 6x_5 = 9 \\ x_1 + 3x_2 + 4x_3 + x_4 + 3x_5 = 7 \\ 2x_1 + 2x_2 + 8x_3 + x_4 + 3x_5 = 10 \end{cases} \quad \text{or } A = \begin{pmatrix} 2 & 1 & 8 & 1 & 6 \\ 1 & 3 & 4 & 1 & 3 \\ 2 & 2 & 8 & 1 & 3 \end{pmatrix}, \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \vec{b} = \begin{pmatrix} 9 \\ 7 \\ 10 \end{pmatrix}.$$

Expected: (a) Augmented matrix and Toolkit steps to the reduced row echelon form (RREF). (b) Translation of RREF to scalar equations. Identify lead and free variables. (c) Scalar general solution for x_1 to x_5. (d) Vector formula for the solution of $A\vec{u} = \vec{b}$. (e) Strang's special solutions for $A\vec{u} = \vec{0}$.

(a) [20%] Find the augmented matrix C and display the toolkit steps to the reduced row echelon form of C. $[A \mid b]$

$$\left[\begin{array}{ccccc|c} 2 & 1 & 8 & 1 & 6 & 9 \\ 1 & 3 & 4 & 1 & 3 & 7 \\ 2 & 2 & 8 & 1 & 3 & 10 \end{array}\right] \underset{\sim}{\text{swap}(1,2)} \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 1 & 3 & 7 \\ 2 & 1 & 8 & 1 & 6 & 9 \\ 2 & 2 & 8 & 1 & 3 & 10 \end{array}\right] \begin{array}{c} \text{combo}(1,2,-2) \\ \sim \\ \text{combo}(1,3,-2) \end{array} \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 1 & 3 & 7 \\ 0 & -5 & 0 & -1 & 0 & -5 \\ 0 & -4 & 0 & -1 & -3 & -4 \end{array}\right] \underset{\sim}{\text{mult}(2,-\tfrac{1}{5})} \left[\begin{array}{ccccc|c} 1 & 3 & 4 & 1 & 3 & 7 \\ 0 & 1 & 0 & 1/5 & 0 & 1 \\ 0 & -4 & 0 & -1 & -3 & -4 \end{array}\right]$$

$$\begin{array}{c} \text{combo}(2,1,-3) \\ \sim \\ \text{combo}(2,3,4) \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 4 & 2/5 & 3 & 4 \\ 0 & 1 & 0 & 1/5 & 0 & 1 \\ 0 & 0 & 0 & -1/5 & -3 & 0 \end{array}\right] \underset{\sim}{\text{mult}(3,-5)} \left[\begin{array}{ccccc|c} 1 & 0 & 4 & 2/5 & 3 & 4 \\ 0 & 1 & 0 & 1/5 & 0 & 1 \\ 0 & 0 & 0 & 1 & 15 & 0 \end{array}\right] \begin{array}{c} \text{combo}(3,1,-2/5) \\ \sim \\ \text{combo}(3,2,-1/5) \end{array} \left[\begin{array}{ccccc|c} 1 & 0 & 4 & 0 & -3 & 4 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 15 & 0 \end{array}\right]$$

$$\text{rref}(C) = \left[\begin{array}{ccccc|c} 1 & 0 & 4 & 0 & -3 & 4 \\ 0 & 1 & 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & 1 & 15 & 0 \end{array}\right]$$

(b) [20%] Write the **scalar equations** corresponding to **rref**(C). Then identify the **free** variables and the **lead** variables.

$x_1 + 4x_3 - 3x_5 = 4$

$x_2 - 3x_5 = 1$

$x_4 + 15x_5 = 0$

free variables: x_3, x_5

lead variables: x_1, x_2, x_4

(c) [20%] Display the **scalar** general solution.

$$x_1 = 4 - 4t_1 + 3t_2$$
$$x_2 = 1 + 3t_2$$
$$x_3 = t_1$$
$$x_4 = -15t_2$$
$$x_5 = t_2$$

(d) [20%] Extract from the answer in (c) a **vector formula** for $\vec{\mathbf{u}}$, the general solution of $A\vec{\mathbf{u}} = \vec{\mathbf{b}}$.

$$\bar{u} = \underbrace{\begin{bmatrix} 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}_p} + \underbrace{t_1 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 3 \\ 3 \\ 0 \\ -15 \\ 1 \end{bmatrix}}_{\bar{u}_h}$$

(e) [20%] Extract from the answer in (d) a vector solution basis for the homogeneous problem $A\vec{\mathbf{u}} = \vec{\mathbf{0}}$. These vectors are called **Strang's Special Solutions**.

$$\left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \\ 6 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 0 \\ -15 \\ 1 \end{bmatrix} \right\}$$

* vectors which compose the homogeneous solution

2. (Chapters 2 and 3: 100 points) Inverse, Determinant, Cramer's Rule.

(a) [30%] Let B be an invertible matrix. Compute the determinant of $(B^T B)^{-1}$ in terms of the determinant of B.

A

$$|(B^TB)^{-1}| = |(B^T)^{-1}||B^{-1}| = |(B^{-1})^T||B^{-1}| = |B^{-1}||B^{-1}| = \frac{1}{|B|^2}$$

(b) [40%] Find the inverse of the matrix $A = \begin{pmatrix} 1 & 5 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$[A|I]$

A

$$\left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 1 & 6 & 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\text{combo}(1,2,-1)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 5 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{\text{combo}(2,1,-5)}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -5 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 6 & -5 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$A^{-1}A = \begin{bmatrix} 6 & -5 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 5 & 0 \\ 1 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

(c) [30%] Let P, Q, R be undisclosed real numbers. Define matrix C and vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{b}}$ by the equations

A

$$C = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{\mathbf{b}} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $C\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

$$X_3 = \frac{|C_3(b)|}{|C|} = \frac{\begin{vmatrix} -2 & 0 & P \\ 0 & -2 & Q \\ 2 & 1 & R \end{vmatrix}}{\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 1 \\ 2 & 1 & 2 \end{vmatrix}} \begin{matrix} \text{cofactor exp. along col 1} \\ = \\ \text{cofactor exp. along row 1} \end{matrix} \frac{-2(-2R-Q)+2(2P)}{-2(-4-1)}$$

$$= \frac{4R+2Q+4P}{10} = \boxed{\frac{2R+Q+2P}{5}}$$

3. (Chapters 1 to 4: 100 points) Independence Tests for Functions.

Theorem (Wronskian test). Wronskian determinant of f_1, f_2, f_3 nonzero at some invented $x = x_0$ implies independence of f_1, f_2, f_3.

Theorem (Sampling test). Functions f_1, f_2, f_3 are independent if a sampling matrix constructed for some invented samples x_1, x_2, x_3 has nonzero determinant.

Let V be the vector space of all functions on $(-\infty, \infty)$. Define functions in V by the equations $f_1(x) = x^3$, $f_2(x) = x$, $f_3(x) = 1 + x$.

$$|\,| = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix}$$

A

(a) [50%] Construct the Wronskian matrix W of the given functions f_1, f_2, f_3, then invent a value for x such that $|W| \neq 0$.

$$|W| = \begin{vmatrix} x^3 & x & 1+x \\ 3x^2 & 1 & 1 \\ 6x & 0 & 0 \end{vmatrix}$$

*use $x = 1$:

$$|W(1)| = \begin{vmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 6 & 0 & 0 \end{vmatrix} \overset{\text{cofactor exp. along rows}}{=} 6(1-2) = -6 \neq$$

A

(b) [50%] Construct a sampling matrix S for the given functions f_1, f_2, f_3 such that $|S| \neq 0$.

$$|S| = \begin{vmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{vmatrix}$$

$x = 0, 1, 2$

$$|S| = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 2 \\ 8 & 2 & 3 \end{vmatrix} \overset{\text{cofactor exp. along row 1}}{=} 1(2-8) = -6 \neq 0$$

4. (Chapters 1 to 4: 100 points) Independence Tests for Column Vectors.

Rank test	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.
Determinant test	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.
Pivot test	Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.
Orthogonality test	A set of nonzero pairwise orthogonal vectors is independent.

Let V be the vector space of all functions on $(-\infty, \infty)$. It is known that functions $1 + e^x$, $x - e^x$, $x + e^x$ are independent in V. Let $S = \textbf{span}(1+e^x, x-e^x, x+e^x)$. Define a coordinate map isomorphism from S to $\mathcal{R}^3$ by

$$T : c_1(1+e^x) + c_2(x-e^x) + c_3(x+e^x) \quad \text{maps into} \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(a) [60%] Define functions in V by the equations $f_1(x) = 2x + 3e^x$, $f_2(x) = 3x + 4e^x$, $f_3(x) = 1 + x + e^x$. Calculate the column vectors $\vec{v}_1 = T(f_1), \vec{v}_2 = T(f_2), \vec{v}_3 = T(f_3)$.

V_1:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 1 & -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 2 & 5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 5/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 5/2 \end{array}\right]$$

$$\boxed{\bar{V}_1 = \begin{bmatrix} 0 \\ -1/2 \\ 5/2 \end{bmatrix}}$$

V_2:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 1 & -1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & 1 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 7/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 1 & 7/2 \end{array}\right]$$

$$\boxed{\bar{V}_2 = \begin{bmatrix} 0 \\ -1/2 \\ 7/2 \end{bmatrix}}$$

3:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 1/2 \end{array}\right]$$

$$\boxed{\bar{V}_3 = \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix}}$$

(b) [40%] Because T is one-to-one and onto, then the given functions f_1, f_2, f_3 are independent in S if and only if the column vectors $T(f_1), T(f_2), T(f_3)$ are independent in $\mathcal{R}^3$. Show details for **one** of the independence tests cited above applied to the column vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ calculated above in part (a).

DETERMINANT TEST - can be applied, have square matrix

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -1/2 & -1/2 & 1/2 \\ 5/2 & 7/2 & 1/2 \end{bmatrix}$$

$$|V| = \begin{vmatrix} 0 & 0 & 1 \\ -1/2 & -1/2 & 1/2 \\ 5/2 & 7/2 & 1/2 \end{vmatrix} \underset{\text{along row 1:}}{\overset{\text{cofactor exp.}}{=}} 1\left(-\tfrac{1}{2}\left(\tfrac{7}{2}\right) - \left(-\tfrac{1}{2}\right)\left(\tfrac{5}{2}\right)\right) = \left(-\tfrac{7}{4} + \tfrac{5}{4}\right) = -\tfrac{1}{2} \neq 0$$

$|V| \neq 0$, so v_1, v_2, v_3 are independent.

5. (Chapters 2, 4 and 6: 100 points) Subspace Definition and Theorems.

DEFINITION. Subset S of vector space V is a subspace of V provided (1), (2), (3) hold:

(1) S contains vector $\vec{0}$.

(2) If $\vec{x}$ and $\vec{y}$ are in S, then $\vec{x}+\vec{y}$ is in S.

(3) If c is a constant and $\vec{x}$ is in S, then $c\vec{x}$ is in S.

(a) [50%] Let V be the vector space of all 2×2 matrices. Let subset S be all vectors $\vec{v} = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$ with a, b, c exhausting all possible constants. The **span theorem** will be applied to show that S is a subspace of V. **Your task**: define vectors (i.e., 2×2 matrices) $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in S such that $S = \mathbf{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

$\vec{V}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $\vec{V}_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ $\vec{V}_3 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\text{Span}(V_1, V_2, V_3) = C_1V_1 + C_2V_2 + C_3V_3 = C_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + C_2\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$
$+ C_3\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}$
$= \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = S$

$\Rightarrow S = \text{Span}(V_1, V_2, V_3)$

$\Rightarrow S$ is a subspace of V

(b) [50%] Let V be the vector space of all continuous functions $f(x)$ defined on $0 \le x \le 1$. Let S be the set of all functions $f(x)$ in V such that $\int_0^1 f(x)dx = 0$. Supply proof details which verify that S is a subspace of V. Cite a theorem or else verify conditions (1), (2), (3) in the DEFINITION.

Verify Definition:

(1) $\vec{0}$ is in S

Set $f(x)=0$ $\quad \int_0^1 f(x)dx = \int_0^1 0\,dx = 0 \Rightarrow$ in V

(2) Given V_1, V_2 in V, $V_1 + V_2$ must be in V

$V_1 = f(x)$ Given: $\int_0^1 f(x)\,dx = 0$

$V_2 = g(x)$ Given: $\int_0^1 g(x)\,dx = 0$

$V_1 + V_2 \Rightarrow \int_0^1 (f(x)+g(x))dx = \int_0^1 f(x)dx + \int_0^1 g(x)\,dx$
$= 0 + 0 = 0 \Rightarrow$ in V

(3) Given V_1, any constant c times V_1 must also be in V.

$V_1 = f(x)$ Given: $\int_0^1 f(x)dx = 0$

$cV_1 \Rightarrow \int_0^1 c\,f(x)dx = c\int_0^1 f(x)dx = c(0) = 0$
$\Rightarrow$ still in V

6. (Chapter 6: 100 points) Gram-Schmidt Orthogonalization Process.

Let S be the subspace of $\mathcal{R}^4$ spanned by the columns $\vec{\mathbf{x}}_1, \vec{\mathbf{x}}_2, \vec{\mathbf{x}}_3$ of the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Find a Gram-Schmidt orthogonal basis $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ for subspace S.

A

$$\vec{V}_1 = x_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{V}_2 = X_2 - \frac{X_2 \cdot V_1}{V_1 \cdot V_1} V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 0V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{V}_3 = X_3 - \frac{X_3 \cdot V_2}{V_2 \cdot V_2} V_2 - \frac{X_3 \cdot V_1}{V_1 \cdot V_1} V_1$$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{1} V_2 - \frac{1}{1} V_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$S = \text{span}(V_1, V_2, V_3) = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right)$$

7. (Chapters 1 to 6: 100 points) Symmetric Matrices. Invertible Matrix Theorem.

(a) [40%] Let A and B be two 20×20 matrices. Prove that $C = A^T A + B + B^T$ is **symmetric**, that is, $C^T = C$.

$C = A^T A + B + B^T$

$C^T = (A^T A + B + B^T)^T$

A

$= (A^T A)^T + (B)^T + (B^T)^T$

$= A^T A + B^T + B$

$= A^T A + B + B^T$ $\square$

(b) [60%] Let A be a 20×8 matrix. Assume that 8×8 matrix $A^T A$ has independent columns. Prove that **rank**$(A) = 8$.

Expected: A referenced result from *"The Invertible Matrix Theorem"* should appear as a fully stated LEMMA, the proof of the LEMMA deferred to the textbook.

A. Lemma: If $A^T A$ has independent cols, A has independent cols

Pf: $A\vec{x} = 0$

$A^T A \vec{x} = 0 \Rightarrow \vec{x}$ must be zero if $A^T A$ has ind cols

$\vec{x}$ must be zero for $A\vec{x} = 0$ as well

$\therefore A$ has independent cols

The rank is the number of pivot columns. Since A has linearly independent cols by Lemma, it will have 8 pivots. $\therefore$ There will be 8 pivot columns and the rank$(A) = 8$

$\square$

8. (Chapter 5: 100 points) Eigenanalysis and Diagonalization.

Definition: An **eigenpair** $(\lambda, \vec{v})$ of A is determined by the equation $A\vec{v} = \lambda\vec{v}$, where λ is a real or complex number and $\vec{v} \neq \vec{0}$.
Expected: To compute eigenpair $(\lambda, \vec{v})$: (1) Compute the RREF of $A - \lambda I$, (2) Compute all eigenvectors for λ as Strang's solutions.

A

(a) [20%] Is $\left(1, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right)$ an eigenpair of the matrix $\begin{pmatrix} 1&1&1 \\ 1&0&0 \\ 1&0&2 \end{pmatrix}$? Answer YES or NO without computing eigenvalues or eigenvectors.

$$\begin{bmatrix} 1&1&1 \\ 1&0&0 \\ 1&0&2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$\Rightarrow \left(1, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right)$ is an eigenpair of the matrix

(b) [40%] The eigenvalues of matrix $A = \begin{pmatrix} 12 & 4 & -1 \\ -4 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$ are 2, 8 and 8. Compute the eigenpairs of A for $\lambda = 8$.

A

$(A - 8I)\vec{x} = \vec{0}$

combo(1,2,1) combo(2,3,-2)

$$\left[\begin{array}{ccc|c} 4 & 4 & -1 & 0 \\ -4 & -4 & -2 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right] \approx \left[\begin{array}{ccc|c} 4 & 4 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right] \approx \left[\begin{array}{ccc|c} 4 & 4 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

mult(2, $-\frac{1}{3}$) combo(2,1,1) mult(1, 1/4)

$$\approx \left[\begin{array}{ccc|c} 4 & 4 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \approx \left[\begin{array}{ccc|c} 4 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_2$
$x_2 = \text{free}$ $\Rightarrow$ Eigenpair for $\lambda = 8$: $\left(8, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)$
$x_3 = 0$ $\quad v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

(c) [40%] (a) [40%] Define matrix $B = \begin{pmatrix} 12 & 4 & -1 \\ -4 & 4 & -2 \\ 0 & 0 & 8 \end{pmatrix}$. Matrix B has eigenvalues 8, 8, 8.

Is matrix B diagonalizable? Explain why or why not.

A

Find eigenvectors w/ $\lambda = 8$

$\Rightarrow (B - 8I)\vec{x} = \vec{0}$

$$\left[\begin{array}{ccc|c} 4 & 4 & -1 & 0 \\ -4 & -4 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \approx \left[\begin{array}{ccc|c} 4 & 4 & -1 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = -x_2$

$x_2 =$ free

$x_3 = 0$

Eigenpairs: $\left(8, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right)$

$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

B is not diagonalizable because it has a repeating eigenvalue of 8, which only has one corresponding eigenvector. In order for a matrix to be diagonalizable, it must have 3 full eigenpairs.

Thus, since B only has one eigenpair, it is not diagonalizable.

9. (Chapter 6: 100 points) Near Point Theorem and Least Squares.

(a) [60%] Define $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Write the normal equations for the inconsistent problem $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$ and solve for the least squares solution $\hat{\mathbf{X}}$.

$$A^T A \vec{x} = A^T \vec{b} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

mult(1, 1/2), mult(2, 1/3)

$$\Rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix} \quad \text{Solve: } \left[\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & 3 & 6 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 0 & -1/2 \\ 0 & 1 & 2 \end{array}\right]$$

A

$$\hat{X} = \begin{bmatrix} -1/2 \\ 2 \end{bmatrix}$$

(b) [20%] Continue part (a). Compute vector $\hat{\mathbf{B}} = A\hat{\mathbf{x}}$, which is the near point to $\vec{\mathbf{b}}$ in the column space of A.

A

$$\hat{B} = A\hat{x} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1/2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ 2 \\ 3/2 \end{bmatrix}$$

(c) [20%] Least squares can be used to find the best fit line $y = mx + b$ for the (x, y)-data points $(-1, 2)$, $(0, 3)$, $(1, 1)$. Show how to change the data into a matrix problem $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$. Then find the line equation by the method of least squares.

Expected: The matrix A you create for part (c) should match the matrix A of part (a). Save time by using the computations from (a) and (b).

$$A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

Based on Solution from (a):

$$\vec{x} = \begin{bmatrix} -1/2 \\ 2 \end{bmatrix}$$

Line equation: $y = -\frac{1}{2}x + 2$

10. (Chapter 7: 100 points) Spectral Theorem and $AQ = QD$.

The spectral theorem says that a symmetric matrix A satisfies $AQ = QD$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 5 & 2 \\ 2 & 5 \end{pmatrix}$.

A $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} = 0 \qquad (5-\lambda)^2 - 4 = 0$$

$$\lambda = 7, 3$$

$(A - 7I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 2 & -2 & 0 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = x_2 \\ x_2 = \text{free} \end{array} \quad \begin{array}{l} V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ V_1 = \frac{V_1}{\|V_1\|} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \end{array}$$

$(A - 3I)\vec{x} = \vec{0}$

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -x_2 \\ x_2 = \text{free} \end{array} \quad \begin{array}{l} V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ V_2 = \frac{V_2}{\|V_2\|} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \end{array}$$

$$Q = \left[V_1 \mid V_2 \right] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix}$$

$u_1 = \frac{Av_1}{\sigma_1}$

11. (Chapter 7: 100 points) Singular Value Decomposition.

Determine the singular value decomposition matrices U, Σ and V for the matrix

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 7 \end{pmatrix}.$$

A

$$A^T A = \begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 51 & 28 \\ 28 & 51 \end{bmatrix}$$

$|A^T A - \lambda I| = 0$

$$\begin{vmatrix} 51-\lambda & 28 \\ 28 & 51-\lambda \end{vmatrix} = 0 \qquad (51-\lambda)^2 - 28^2 = 0$$

$\lambda = 51 \pm 28 = 79, 23$

Thus: $\sigma = \sqrt{79}, \sqrt{23}$

$$(A^T A - 79 I)\vec{x} = \vec{0} \quad \left[\begin{array}{cc|c} -28 & 28 & 0 \\ 28 & -28 & 0 \end{array}\right] \approx \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = x_2 \\ x_2 = \text{free} \end{array} \quad v_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$(A^T A - 23 I)\vec{x} = \vec{0} \quad \begin{bmatrix} 28 & 28 \\ 28 & 28 \end{bmatrix} \approx \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} x_1 = -x_2 \\ x_2 = \text{free} \end{array} \quad v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$$

$$u_1 = \frac{Av_1}{\sigma_1} = \frac{\begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{\sqrt{79}} = \frac{\begin{bmatrix} 9 \\ 9 \end{bmatrix}}{\sqrt{158}} = \begin{bmatrix} 9/\sqrt{158} \\ 9/\sqrt{158} \end{bmatrix}$$

$$u_2 = \frac{Av_2}{\sigma_2} = \frac{\begin{bmatrix} 7 & 2 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}}{\sqrt{23}} = \frac{\begin{bmatrix} -5 \\ 5 \end{bmatrix}}{\sqrt{46}} = \begin{bmatrix} -5/\sqrt{46} \\ 5/\sqrt{46} \end{bmatrix}$$

$$U = \left[u_1 \,|\, u_2 \right] = \begin{bmatrix} 9/\sqrt{158} & -5/\sqrt{46} \\ 9/\sqrt{158} & 5/\sqrt{46} \end{bmatrix} \qquad \Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} = \begin{bmatrix} \sqrt{79} & 0 \\ 0 & \sqrt{23} \end{bmatrix}$$

$$V = \left[v_1 \,|\, v_2 \right] = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}.$$

12. (Chapters 4 and 6: 100 points) Linear Transformations. Orthogonality.

(a) [40%] Let the linear transformation T from $\mathcal{R}^2$ to $\mathcal{R}^2$ be defined by its action on two independent vectors:

A

$$T\left(\begin{pmatrix}2\\1\end{pmatrix}\right)=\begin{pmatrix}4\\3\end{pmatrix},\quad T\left(\begin{pmatrix}1\\3\end{pmatrix}\right)=\begin{pmatrix}4\\1\end{pmatrix}.$$

Find the unique 2×2 matrix A such that T is defined by the matrix multiply equation $T(\vec{\mathbf{x}})=A\vec{\mathbf{x}}$

$$A=\begin{pmatrix}a & b\\ c & d\end{pmatrix}\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}=\begin{pmatrix}4\\3\end{pmatrix}\qquad \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}$$

$$\begin{cases}2a+b=4\\ a+3b=4\\ 2c+d=3\\ c+3d=1\end{cases}\qquad \begin{pmatrix}2&1&0&0\\1&3&0&0\\0&0&2&1\\0&0&1&3\end{pmatrix}\begin{pmatrix}a\\b\\c\\d\end{pmatrix}=\begin{pmatrix}4\\4\\3\\1\end{pmatrix}$$

$$\begin{pmatrix}2&1&0&0&4\\1&3&0&0&4\\0&0&2&1&3\\0&0&1&3&1\end{pmatrix}\sim\begin{pmatrix}1&-2&0&0&0\\1&3&0&0&4\\0&0&1&-2&2\\0&0&1&3&1\end{pmatrix}\begin{matrix}\text{combo}(2,1,-1)\\ \text{combo}(4,3,-1)\end{matrix}\sim\begin{pmatrix}1&-2&0&0&0\\0&5&0&0&4\\0&0&1&-2&2\\0&0&0&5&-1\end{pmatrix}\begin{matrix}\text{combo}(1,2,-1)\\ \text{combo}(3,4,-1)\end{matrix}$$

$$\begin{pmatrix}1&-2&0&0&0\\0&1&0&0&4/5\\0&0&1&-2&2\\0&0&0&1&-1/5\end{pmatrix}\begin{matrix}\text{mult}(2,1/5)\\ \text{mult}(4,1/5)\end{matrix}\sim\begin{pmatrix}1&0&0&0&8/5\\0&1&0&0&4/5\\0&0&1&0&8/5\\0&0&0&1&-1/5\end{pmatrix}\quad\begin{cases}a=8/5\\ b=4/5\\ c=8/5\\ d=-1/5\end{cases}$$

$$\boxed{A=\begin{pmatrix}8/5 & 4/5\\ 8/5 & -1/5\end{pmatrix}}$$

(b) [60%] Let W be the subspace of $\mathcal{R}^4$ spanned by the columns of the 4×2 matrix

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

Compute a basis for $W^\perp$, the subspace of $\mathcal{R}^4$ orthogonal to W.

> **Expected**: Use this corollary to the fundamental theorem of linear algebra:
> **THEOREM**. If W is the column space of matrix A, then $W^\perp$ is the nullspace of A^T.

A

A basis for $W^\perp$ must be a basis for Null (A^T) by the Fundamental Theorem of Linear Algebr

Find $A^T\vec{x} = \vec{0}$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}\vec{x} = \vec{0}$$

$$\begin{cases} x_1 = -t_1 \\ x_2 = t_1 \\ x_3 = -t_2 \\ x_4 = t_2 \end{cases} \qquad t_1\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}.$$

The basis for Null (A^T) is the set of Strang's solutions to $A^T\vec{x} = \vec{0}$

$$\boxed{B_{W^\perp} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}}$$

13. (Chapters 1 to 7: 100 points) Fundamental Theorem of Linear Algebra.

(a) [40%] Define concisely the four fundamental subspaces appearing in the Fundamental Theorem of Linear Algebra.

A

Col(A) = Columnspace of A. Has a basis made up of the column vectors of A.

Row(A) = rowspace of A. Has a basis made up of the column vectors of A^T.

Null(A) = Nullspace of A. Has a basis made up of Strang's special sols to the equation $A\vec{x} = \vec{0}$

Null(A^T) = Nullspace of A^T. Has a basis made up of Strang's special sols to the equation $A^T\vec{x} = \vec{0}$

(b) [60%] Fill in the **nine** (9) boxes below that have a question mark **(?)**. **Details:**

A

(1) In each of matrices U, Σ and V, fill in the missing fundamental subspace name which occupies the corresponding columns. The column space of A is already filled for the first r columns of U. Symbol r is the number of nonzero singular values.

(2) Matrix A is $m \times n$. Fill in the size of the matrices U, Σ, V.

(3) Fill in the column counts for each of the four subspaces. The column space of A is spanned by the first r columns, which is already filled into the figure.

Size $= m \times n$ $\quad A = U \Sigma V^T$ Singular Value Decomposition

Size = $\boxed{M \times M ?}$ $\quad U =$

colspace(A)	Nullspace(A^T) ?
No. columns $= r$	columns = $\boxed{r - M\ ?}$

Size = $\boxed{M \times n\ ?}$ $\quad \Sigma =$

$$\left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ & \vdots & & \mathbf{0} \\ 0 & \cdots & \sigma_r & \\ \hline & \mathbf{0} & & \mathbf{0} \end{array}\right]$$

Size = $\boxed{n \times n\ ?}$ $\quad V =$

Rowspace(A) ?	Nullspace(A) ?
No. columns = $\boxed{r\ ?}$	No. columns = $\boxed{r - n\ ?}$