MATH 2270-4 Sample Exam 3 S2019

Please, no books, notes or electronic devices.

Some questions involve proofs. Please divide your time accordingly.

Extra details can appear on the back side or on extra pages. Please supply a road map for details not directly following the problem statement.

Details count 75% and answers count 25%. Exam 3 on April 12 will have 6 questions, obtained from this sample by deleting two questions and editing the content below. Expect the April 12 exam to be like this sample, problems edited, but no change in background required.

Problem 1. **(100 points)** Find the complete vector solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Expected: (a) Augmented matrix. (b) Toolkit steps to RREF. (c) Translation of RREF to scalar equations. (d) Scalar general solution. (e) Find the homogeneous solution $\vec{x}_h$, which is a linear combination of Strang's special solutions. (f) Find a particular solution $\vec{x}_p$. (g) Write the vector general solution $\vec{\mathbf{x}} = \vec{\mathbf{x}}_h + \vec{\mathbf{x}}_p$.

Definition: An *abstract* vector space V is a data set of packages called **vectors** together with operations of addition $(+)$ and scalar multiplication $(\cdot)$ satisfying the following eight (8) rules:

(1) If $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in V, then $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is defined and in V.

(2) $\vec{\mathbf{x}} + (\vec{\mathbf{y}} + \vec{\mathbf{z}}) = (\vec{\mathbf{x}} + \vec{\mathbf{y}}) + \vec{\mathbf{z}}$

(3) There is a zero vector $\vec{\mathbf{0}}$ with $\vec{\mathbf{x}} + \vec{\mathbf{0}} = \vec{\mathbf{x}}$.

(4) $\vec{\mathbf{x}} + \vec{\mathbf{y}} = \vec{\mathbf{y}} + \vec{\mathbf{x}}$

(5) If $c =$ constant and $\vec{\mathbf{x}}$ is in V, then $c \cdot \vec{\mathbf{x}}$ is defined and in V.

(6) $(a + b) \cdot \vec{\mathbf{x}} = a \cdot \vec{\mathbf{x}} + b \cdot \vec{\mathbf{x}}$

(7) $(ab) \cdot \vec{\mathbf{x}} = a \cdot (b \cdot \vec{\mathbf{x}})$

(8) $1 \cdot \vec{\mathbf{x}} = \vec{\mathbf{x}}$

Definition. If vectors $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$ are a basis for subspace X of an *abstract* vector space V, and $\vec{\mathbf{x}} = c_1\vec{\mathbf{b}}_1 + c_2\vec{\mathbf{b}}_2 + c_3\vec{\mathbf{b}}_3$ is a given linear combination of these vectors, then the uniquely determined constants c_1, c_2, c_3 are called the *coordinates of* $\vec{\mathbf{x}}$ *relative to the basis* $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$ and the *coordinate map* is the isomorphism

$$\vec{\mathbf{x}} = c_1\vec{\mathbf{b}}_1 + c_2\vec{\mathbf{b}}_2 + c_3\vec{\mathbf{b}}_3 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Problem 2. (100 points) Let V be the vector space of all functions on $(-\infty, \infty)$. Define $W = \mathbf{span}\{1, x, x^2\}$. Assume known that $1, x, x^2$ are independent functions. Define subspace $S = \mathbf{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

$$\vec{v}_1 : \ y = 1 + x, \quad \vec{v}_2 : \ y = 2 + x^2, \quad \vec{v}_3 : \ y = 2 + x + x^2.$$

(a) [20%] Explain why S is contained in W, that is, provide details for why vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are in W.

(b) [40%] Prove that $W = S$. Therefore $\dim(S) = \dim(W) = 3$, which proves that $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are independent.

(c) [40%] Define vector $\vec{\mathbf{v}}$ in S by equation $y = 3 + 4x + x^2$. **Compute** c_1, c_2, c_3 satisfying the equation $\vec{\mathbf{v}} = c_1\vec{\mathbf{v}}1 + c_2\vec{\mathbf{v}}_2 + c_3\vec{\mathbf{v}}_3$, using coordinate map methods.

Expected in (c): Calculations of c_1, c_2, c_3 are to be done using column vectors from $\mathcal{R}^3$, not functions from V. **Zero credit** for not using column vectors.
Definition: A subset S of a vector space V is a **subspace** of V provided
(1) The zero vector is in S
(2) If vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{y}}$ are in S, then $\vec{\mathbf{x}} + \vec{\mathbf{y}}$ is in S.
(3) If vector $\vec{\mathbf{x}}$ is in S and c is any scalar, then $c\vec{\mathbf{x}}$ is in S.

Problem 3. (100 points)
(a) [40%] Let V be the vector space $\mathcal{R}^4$. Invent an example of a non-void subset S of V that satisfies (1) and (2) but fails the third item (3).

(b) [60%] Let V be an *abstract* vector space. Let $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ be three vectors in V. Define S to be the set of all linear combinations of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. Prove:

SPAN THEOREM. Subset S is a subspace of V.

Expected: A proof uses only the symbols $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ and the 8 rules. Column vectors cannot be used in this proof. Please don't try to burst packages $\vec{\mathbf{x}}$.

Problem 4. (100 points) Let A be a 4×3 matrix. Assume the columns of A^TA are independent. Prove or disprove that A has independent columns.

Expected: To prove the claim, assemble details and theorem citations to support the claim. To disprove the claim, invent a specific 4×3 matrix that violates the claim.

Problem 5. (100 points) Let U be a 2×2 matrix with $U^TU = I$. Let $\vec{\mathbf{u}}_1, \vec{\mathbf{u}}_2$ denote the columns of U. Prove that the columns of U are orthonormal.

Problem 6. (100 points) Find the orthogonal projection vector $\vec{\mathbf{v}}$ (the shadow projection vector) of $\vec{\mathbf{y}}$ onto the direction of $\vec{\mathbf{u}}$, given

$$\vec{\mathbf{y}} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{u}} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Problem 7. (100 points) Let W be the column space of $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$ and let $\vec{\mathbf{b}} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

(a) [40%] Define the **normal equations** for the problem $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$.

(b) [60%] Let $\vec{\mathbf{N}}$ be the near point to $\vec{\mathbf{b}}$ in the subspace W. Find $\vec{\mathbf{N}}$.

Problem 8. (100 points) The **Fundamental Theorem of Linear Algebra** contains this statement: The row space of a matrix is orthogonal to the null space of the matrix. This means that $\vec{\mathbf{R}} \cdot \vec{\mathbf{N}} = 0$ for each vector $\vec{\mathbf{R}}$ in the row space and each vector $\vec{\mathbf{N}}$ in the null space.

Let A be an $m \times n$ matrix. Define subspaces S_1 = column space of A, S_2 = null space of A^T. Give a direct proof (using no textbook theorems) that the only vector $\vec{v}$ in both S_1 and S_2 is the zero vector.