Problem 1. **(100 points)** Define matrix A, vector $\vec{\mathbf{b}}$ and vector variable $\vec{\mathbf{x}}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}, \quad \vec{\mathbf{b}} = \begin{pmatrix} 3 \\ -5 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

A

(a) [40%] For the system $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, display the **formula** for x_2 according to Cramer's Rule. **Don't compute** x_2! **Don't expand determinants!**

$$X_2 = \frac{|A_2(b)|}{|A|} = \frac{\begin{vmatrix} -2 & 3 & 0 & 0 \\ 0 & -5 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix}}{\begin{vmatrix} -2 & 3 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{vmatrix}}$$

A

(b) [60%] Find the entry in row 3 and column 2 in matrix A^{-1}, by using the adjugate formula for the inverse: $A^{-1} = \dfrac{\textbf{adj}(A)}{|A|}$.

The answer is a fraction. Matrix A is not triangular, but cofactor expansion applies: $|A| = 16$.

$$\mathrm{adj}(A_{3,2}) = \mathrm{cofactor}(A_{2,3}) = -\begin{vmatrix} -2 & 3 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 2 \end{vmatrix} \quad \text{cofactor exp. along row 3} : = -(2)\begin{vmatrix} -2 & 3 \\ 1 & 4 \end{vmatrix}$$

$$= -2(-8-3) = -2(-11) = 22$$

$$A^{-1}_{3,2} = \frac{\mathrm{adj}(A_{2,3})}{|A|} = \frac{22}{16} = \boxed{\frac{11}{8}}$$

Problem 2. **(100 points)** Define matrix $A = \begin{pmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

$$A = \begin{bmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{bmatrix} \underset{E_1}{\overset{\text{combo}(1,2,-3)}{\sim}} \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 8 & 14 & -4 \end{bmatrix} \underset{E_2}{\overset{\text{combo}(1,3,-4)}{\sim}} \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -4 \end{bmatrix}$$

$$\underset{E_3}{\overset{\text{combo}(2,3,2)}{\sim}} \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = U$$

$* \ U = E_3 E_2 E_1 A$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

$L = E_1^{-1} E_2^{-1} E_3^{-1}$

$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{pmatrix}$ $E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

check: $LU = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{bmatrix}$ ✓

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix} \quad U = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}}$$

Problem 3. **(100 points)** Vector space V is the set of all functions on $0 < x < \infty$. Equations $y = 1$, $y = x^2$, $y = x^3$ represent independent vectors $\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3$ in V and $S = \mathbf{span}\{\vec{\mathbf{b}}_1, \vec{\mathbf{b}}_2, \vec{\mathbf{b}}_3\}$ is a subspace of V. The **coordinate map** T from S to $\mathcal{R}^3$ is defined by

$$c_1\vec{\mathbf{b}}_1 + c_2\vec{\mathbf{b}}_2 + c_3\vec{\mathbf{b}}_3 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \quad \text{or} \quad c_1 + c_2x^2 + c_3x^3 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Define vectors in subspace S:

$$\vec{\mathbf{v}}_1 : \quad y = 1 - x^2, \quad \vec{\mathbf{v}}_2 : \quad y = x^3 - x^2, \quad \vec{\mathbf{v}}_3 : \quad y = 4 + 2x^3.$$

Vectors $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ are mapped by T as follows:

$$1 - x^2 \to \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad x^3 - x^2 \to \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad 4 + 2x^3 \to \begin{pmatrix} 4 \\ 0 \\ 2 \end{pmatrix}.$$

The coordinate map T, an isomorphism, maps **independent sets to independent sets**. Therefore, the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is independent in V if and only if the three column vectors above are independent in $\mathcal{R}^3$.

Apply each of the three independence tests below to establish independence of $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. **Details are expected**: explain briefly how the test applies. *Zero credit for no explanation.*

The phrase **augmented matrix** *used below means the* 3×3 *matrix* $\langle \vec{\mathbf{v}}_1 | \vec{\mathbf{v}}_2 | \vec{\mathbf{v}}_3 \rangle$.

Problem 4 Continued.

A

Wronskian test. Nonzero Wronskian determinant of f_1, f_2, f_3 at invented value $x = x_0$ implies independence of f_1, f_2, f_3.

$V_1 = f_1 \quad V_2 = f_2 \quad V_3 = f_3 \qquad \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} \neq 0$

Details: applied to functions

$$|W(x)| = \begin{vmatrix} 1-x^2 & x^3-x^2 & 4+2x^3 \\ -2x & 3x^2-2x & 6x^2 \\ -2 & 6x-2 & 12x \end{vmatrix} \Rightarrow |W(1)| = \begin{vmatrix} 0 & 0 & 6 \\ -2 & 1 & 6 \\ -2 & 4 & 12 \end{vmatrix} = 6\begin{vmatrix} -2 & 1 \\ -2 & 4 \end{vmatrix} = 6(-8+2) = -36 \neq 0$$

independent: $c_1\bar{V}_1 + c_2\bar{V}_2 + c_3\bar{V}_3 = 0$ (where $\bar{V}_1, \bar{V}_2, \bar{V}_3$ are functions). To solve this function, make a system of eqns of derivatives, $c_1\bar{V}_1' + c_2V_2' + c_3V_3' = 0$ & $c_1V_1'' + c_2V_2'' + c_3V_3'' = 0$. To solve for c_1, c_2, c_3, if the determinant is not 0, it has an inverse & $c_1 = c_2 = c_3 = 0$, meaning the vectors are independent.

A

Determinant test. Three column vectors are independent if their augmented matrix is square and has nonzero determinant.

$V_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad V_3 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \qquad |\bar{V}_1\ \bar{V}_2\ \bar{V}_3| \neq 0$

Details: applied to fixed vectors

$$\begin{vmatrix} 1 & 0 & 4 \\ -1 & -1 & 0 \\ 0 & 1 & 2 \end{vmatrix} \sim \begin{vmatrix} 1 & 0 & 4 \\ 0 & -1 & 4 \\ 0 & 1 & 2 \end{vmatrix}$$

cof. expansion along column 1: $+1\begin{vmatrix} -1 & 4 \\ 1 & 2 \end{vmatrix} = 1(-2-4) = -6 \neq 0$

independent

If $|\bar{V}_1\ \bar{V}_2\ \bar{V}_3| \neq 0$, the matrix is invertible & by invertiple matrix Thm, the columns of $[\bar{V}_1\ \bar{V}_2\ \bar{V}_3]$ are linearly independent.

A

Pivot test. Three column vectors are independent if their augmented matrix A has 3 pivot columns. fixed vectors

Details: If rref has 3 pivot columns $\rightarrow c_1V_1 + c_2V_2 + c_3V_3 = 0$ IFF $c_1 = 0, c_2 = 0, c_3 = 0$ which shows the 3 vectors are linearly independent

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & -1 & 0 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & -1 & 4 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -4 \\ 0 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

pivot columns

3 piv columns $\Rightarrow$ independent.

Problem 4. (100 points) Matrix $A = \begin{pmatrix} 0 & 4 & -1 \\ 4 & 0 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ has real eigenpairs

$$\left(5, \begin{pmatrix} -1 \\ 1 \\ 9 \end{pmatrix}\right), \quad \left(4, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(-4, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\right).$$

A (a) [30%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

$P =$ columns of eigenvectors $= \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 9 & 0 & 0 \end{bmatrix}$ $\quad D =$ diag. matrix of corresponding eigenvalues $= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

A (b) [20%] Display a symbolic matrix product formula for A in terms of P and D. To save time, **do not evaluate anything**.

$AP = PD$ where P is invertible, so

$$\boxed{A = PDP^{-1}}$$

A|A- (c) [50%] Show the details for computing an eigenvector for $\lambda = 5$. $\rightarrow (A - \lambda I)x = 0$

$\lambda = 5$:

$$[A - \lambda I] = \begin{bmatrix} -5 & 4 & -1 \\ 4 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/5 & 1/5 \\ 4 & -5 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/5 & 1/5 \\ 0 & -9/5 & 1/5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -4/5 & 1/5 \\ 0 & 1 & -1/9 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1/9 \\ 0 & 1 & -1/9 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{matrix} x_1 + 1/9\, x_3 = 0 \\ x_2 - 1/9\, x_3 = 0 \\ x_3 = \text{free} \end{matrix} \Rightarrow \begin{matrix} x_1 = -\frac{1}{9} t_1 \\ x_2 = \frac{1}{9} t_1 \\ x_3 = t_1 \end{matrix} \Rightarrow \bar{x} = \frac{t_1}{9} \begin{bmatrix} -1 \\ 1 \\ 9 \end{bmatrix}$$

eigenvector

$$\left(5, \begin{pmatrix} -1 \\ 1 \\ 9 \end{pmatrix}\right)$$

Problem 5. (100 points)

Definition: A subset S of a vector space V is a **subspace** of V provided

(1) The zero vector is in S.

(2) If vectors $\mathbf{x}$ and $\mathbf{y}$ are in S, then $\mathbf{x}+\mathbf{y}$ is in S.

(3) If vector $\mathbf{x}$ is in S and c is any scalar, then $c\mathbf{x}$ is in S.

Let vector space $V = \mathcal{R}^n$ and let A be a given $m \times n$ matrix.

A (a) [60%] Prove by definition that the equation $A\vec{x} = \vec{0}$ defines a subspace S of V.

Let $\bar{u} \in S. \Rightarrow A\bar{u}=0$. Let $\bar{v} \in S. \Rightarrow A\bar{v}=0$

0 is contained in S because $A(\bar{0})=0$

If $\bar{u}$ & $\bar{v}$ are in S (defined above), $A(\bar{u}+\bar{v}) = A\bar{u}+A\bar{v} = 0+0 = 0$

so $\bar{u}+\bar{v} \in S$.

If $\bar{u}$ is in S (defined above), $A(c\bar{u}) = cA\bar{u} = c(0) = 0$ for any constant $c \in \mathbb{R}$,

so $c\bar{u} \in S$, so S is a subspace of V.

A (b) [40%] Explain why the equation $A\vec{x} = \vec{b}$ fails to define a subspace of V when $\vec{b} \neq \vec{0}$.

If $\bar{b} \neq 0$, Let $A\bar{u} = \bar{b}$, so $\bar{u}$ is in the subset.

$\bar{0}$ is not in the subset because $A(\bar{0}) = 0$, but $\bar{b} \neq \bar{0}$

Since $\bar{0}$ is not in S, S is not a subspace. Fails to meet definition of subspace

(Not a Subspace Thm)

Problem 6. **(100 points)** Let $\vec{\mathbf{v}}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \vec{\mathbf{v}}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^3$ which satisfy the two restriction equations $\vec{\mathbf{v}}_1 \cdot \vec{\mathbf{x}} = 0$, $\vec{\mathbf{v}}_2 \cdot \vec{\mathbf{x}} = 0$. Prove that S is a subspace of $\mathcal{R}^3$.

A

Expected: Cite known theorems, if they apply, to avoid writing a proof. If no theorems are applied, then verify the 3 conditions for the definition of a subspace (see the preceding exam problem).

$V_1 \bar{X} = 0$
$V_2 \bar{X} = 0$ is a set of homogeneous equations so by the kernel theorem, the set is a subspace of $\mathbb{R}^3$.

Let $\bar{u} \in S$. $\Rightarrow \bar{V}_1 \cdot \bar{U} = 0$, $\bar{V}_2 \bar{U} = 0$. Let $\bar{y} \in S$. $\bar{V}_1 \cdot \bar{y} = 0$, $\bar{V}_2 \cdot \bar{y} = 0$

$\bar{0}$ is contained in the subset S because $\bar{V}_1 \cdot \bar{0} = 0$ & $\bar{V}_2 \cdot \bar{0} = 0$

$\bar{u}$ & $\bar{y}$ are in the set S. $\bar{V}_1 \cdot (\bar{u} + \bar{y}) = \bar{V}_1 \cdot \bar{u} + \bar{V}_1 \cdot \bar{y} = 0 + 0 = 0$

& $\bar{V}_2 \cdot (\bar{u} + \bar{y}) = \bar{V}_2 \cdot \bar{u} + \bar{V}_2 \cdot \bar{y} = 0 + 0 = 0$, so $\bar{u} + \bar{y}$ is in S.

$\bar{u}$ is in S, and $\bar{V}_1 \cdot (c\bar{u}) = c(\bar{V}_1 \cdot \bar{u}) = c(0) = 0$ for any constant c, so $c\bar{u}$ is in S. $\therefore$ by definition of a subspace, S is a subspace of $\mathbb{R}^3$.

Problem 7. **(100 points)** Used in this problem are equivalent statements taken from the **Invertible Matrix Theorem**, which says that a square matrix C has an inverse C^{-1} if and only if one of the statements labeled **a** to **x** is true. Three of these statements, for example, are (1) $|C| \neq 0$, (2) C has independent columns, (3) the dimension of the nullspace of C is zero.

A (a) [20%] Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Compute the 3×3 matrix $A^T A$.

$$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

A (b) [80%] Let matrix B be 2×3 with dependent columns. Prove or disprove: The 3×3 matrix $B^T B$ has dependent columns.

proof by contradiction:

Expected: To prove a claim, assemble details and theorem citations to support the claim. To disprove a claim, invent a specific detailed example that violates the claim.

Assume B^TB has independent columns.
B has dependent columns, so $B\bar{x} = 0$ has a nontrivial solution.
multiplying both sides by B^T, $B^T(B\bar{x}) = B^T(\bar{0}) \Rightarrow (B^TB)\bar{x} = 0$.
Because B^TB has independent columns, it is invertible by the Invertible Matrix Theorem (2), so $(B^TB)^{-1}(B^TB)\bar{x} = (B^TB)^{-1}\bar{0}$
$\Rightarrow I\bar{x} = \bar{0} \Rightarrow x = \bar{0}$. However x cannot be the trivial solution since $B\bar{x} = \bar{0}$ has a nontrivial solution.
$\therefore$ By contradiction, B^TB must have dependent columns.