Vector Spaces and Linear Transformations

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Vector Spaces and the Toolkit

Consider any vector model: fixed, free, physics or Gibbs. Let $\boldsymbol{V}$ denote the **data set** of one of these models. The data set consists of packages of data items, called **vectors**. Assume a particular dimension, $\boldsymbol{n}$ for fixed, $\mathbf{2}$ or $\mathbf{3}$ for the others. Let $\boldsymbol{k}$, $\boldsymbol{k_1}$, $\boldsymbol{k_2}$ be constants. Let $\vec{\boldsymbol{X}}$, $\vec{\boldsymbol{Y}}$, $\vec{\boldsymbol{Z}}$ represent three vectors in $\boldsymbol{V}$. The following **toolkit** of eight (8) vector properties can be verified from the definitions.

Closure	The operations $\vec{X} + \vec{Y}$ and $k\vec{X}$ are defined and result in a new data item package [a vector] which is also in V.	
Addition	$\vec{X} + \vec{Y} = \vec{Y} + \vec{X}$	commutative
	$\vec{X} + (\vec{Y} + \vec{Z}) = (\vec{Y} + \vec{X}) + \vec{Z}$	associative
	Vector $\vec{0}$ is defined and $\vec{0} + \vec{X} = \vec{X}$	zero
	Vector $-\vec{X}$ is defined and $\vec{X} + (-\vec{X}) = \vec{0}$	negative
Scalar multiply	$k(\vec{X} + \vec{Y}) = k\vec{X} + k\vec{Y}$	distributive I
	$(k_1 + k_2)\vec{X} = k_1\vec{X} + k_2\vec{X}$	distributive II
	$k_1(k_2\vec{X}) = (k_1k_2)\vec{X}$	distributive III
	$1\vec{X} = \vec{X}$	identity

If you think vectors are arrows, then re-tool your thoughts. Think of vectors as **data item packages**. A technical word, **vector** can also mean a graph, a matrix for a digital photo, a sequence, a signal, an impulse, or a differential equation solution .

Vector Space

Definition 1 (Vector Space)
A data set V equipped with $\boxplus$ and $\boxdot$ operations satisfying the closure law and the eight toolkit properties is called an **abstract vector space**.

The data in V consists of *packages* of data items called **vectors**. The operations $\boxplus$ and $\boxdot$ apply to vectors, that is, to whole packages of data.

The toolkit provides 8 rules to manipulate the packages, without dealing with the inner workings of data items that might be bundled away inside the package. It is usually counterproductive to burst vectors into smaller component pieces. The toolkit hides details and thus encourages data manipulation from its 8 rules.

What's a *Space*?

There is no intended geometrical implication in this term. The usage of **space** originates from phrases like **parking space** and **storage space**.

An **abstract vector space** is a data set for an application, organized as packages of data items called **vectors**, together with $\boxplus$ and $\boxdot$ operations, which satisfy the toolkit of eight manipulation rules.

The packaging of individual data items is structured, or organized by some scheme, which amounts to a *storage space*, hence the term *space*.

What does *abstract* mean?

The technical details of the packaging and the organization of the data set are invisible to the toolkit rules. The toolkit acts on the formal data packages, which are called **vectors**. Briefly, the toolkit is used **abstractly**, devoid of any details of the storage scheme or internal structure of the data set. Implicit in the toolkit rules is *no bursting* of packages allowed.

BASIS of a Vector Space

Definition: A **basis** of a vector space V is a set of vectors $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$ such that every vector $\vec{\mathbf{v}}$ in V can be uniquely written as a linear combination of $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$. Briefly, the vectors *span* V and are *independent*.

Linear Transformation

A linear transformation is a function $\boldsymbol{T}$ defined on a vector space $\boldsymbol{V}$ with range in a vector space $\boldsymbol{W}$ satisfying the rules
(a) $\boldsymbol{T}(\vec{\mathbf{v}}_1 + \vec{\mathbf{v}}_2) = \boldsymbol{T}(\vec{\mathbf{v}}_1) + \boldsymbol{T}(\vec{\mathbf{v}}_2)$
(b) $\boldsymbol{T}(k\vec{\mathbf{v}}_1) = k\boldsymbol{T}(\vec{\mathbf{v}}_1)$.

Theorem 1 (Matrix of T)
Assume $\boldsymbol{V} = \boldsymbol{R}^n$ and $\boldsymbol{W} = \boldsymbol{R}^m$. Then $\boldsymbol{T}$ is represented as a matrix multiply

$$\boldsymbol{T}(\vec{\mathbf{x}}) = \boldsymbol{A}\vec{\mathbf{x}}$$

where $\boldsymbol{A}$ is the $\boldsymbol{n} \times \boldsymbol{m}$ matrix whose columns are given in terms of the identity matrix $\boldsymbol{I}$ and function $\boldsymbol{T}$ by the formula

$$\mathbf{col}(\boldsymbol{A}, \boldsymbol{j}) = \boldsymbol{T}(\mathbf{col}(\boldsymbol{I}, \boldsymbol{j})), \quad \boldsymbol{j} = 1, \ldots, \boldsymbol{n}.$$

Theorem 2 (Representation of T)
Every basis $\{\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n\}$ of $\boldsymbol{V}$ gives a relation

$$\boldsymbol{T}\left(\sum_{j=1}^{n} \boldsymbol{c}_j \vec{\mathbf{v}}_j\right) = \sum_{j=1}^{n} \boldsymbol{c}_j \vec{\mathbf{w}}_j, \quad \text{where} \quad \vec{\mathbf{w}}_j = \boldsymbol{T}(\vec{\mathbf{v}}_j).$$