The Case of Infinitely Many Solutions

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Infinitely Many Solution Case

A system of equations having infinitely many solutions is solved from a toolkit sequence construction that parallels the unique solution case. The same quest for lead variables is made, hoping in the final frame to have just the variable list on the left and numbers on the right.

Stopping Criterion

A toolkit sequence terminates with the final frame, in either the case of a unique solution or infinitely many solutions, with exactly the same criterion:

> The last frame is attained when every nonzero equation has a lead variable. Remaining equations have the form $\mathbf{0} = \mathbf{0}$.

An Illustration: Infinitely Many Solutions

We will solve by toolkit sequence methods, using `swap`, `combo`, `mult`, the following system of equations:

$$\begin{array}{rcrcrcr} & & y & + & 3z & = & 1, \\ x & + & y & & & = & 3, \\ x & + & 2y & + & 3z & = & 4. \end{array}$$

$$\begin{array}{rcrcrcr} & & y & + & 3z & = & 1, \\ x & + & y & & & = & 3, \\ x & + & 2y & + & 3z & = & 4. \end{array}$$

Frame 1. Original system.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ x & + & y & & & = & 3, \\ & & y & + & 3z & = & 1. \end{array}$$

Frame 2.

`swap(1,3)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & y & + & 3z & = & 1. \end{array}$$

Frame 3.

`combo(1,2,-1)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & & & 0 & = & 0. \end{array}$$

Frame 4.

`combo(2,3,1)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & & y & + & 3z & = & 1, \\ & & & & 0 & = & 0. \end{array}$$

Frame 5.

`mult(2,-1)`

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & 2, \\ & & y & + & 3z & = & 1, \\ & & & & 0 & = & 0. \end{array}$$

Frame 6. `combo(2,1,-2)`

Last Frame.

Lead variables x, y.

When to Stop

The last frame is attained when every nonzero equation has a lead variable. Remaining equations have the form $0 = 0$.

$$\begin{array}{rcrcrcl} x & & & - & 3z & = & 2, \\ & & y & + & 3z & = & 1, \\ & & & & 0 & = & 0. \end{array}$$

Last Frame to General Solution

Once the *last frame* of the toolkit sequence is obtained, then the general solution can be written by a fixed and easy-to-learn **last frame algorithm**. This process is used only in case of infinitely many solutions.

(1) Assign invented symbols $\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots$ to the free variables.[a]

(2) Isolate each lead variable.

(3) Back-substitute the free variable invented symbols.

[a] Computer algebra system `maple` uses these invented symbols, hence our convention here is to use $\boldsymbol{t_1}, \boldsymbol{t_2}, \boldsymbol{t_3}, \ldots$ as the list of invented symbols.

Last Frame Algorithm Illustration

From the last frame of the toolkit sequence,

$$\begin{array}{rrrrl} x & & -3z & = & 2, \\ & y & +3z & = & 1, \\ & & 0 & = & 0, \end{array}$$

Last Frame.
Lead variables x, y.

the general solution is written as follows.

$$z = t_1$$

The free variable z is assigned symbol t_1.

$$\begin{aligned} x &= 2 + 3z, \\ y &= 1 - 3z \end{aligned}$$

The lead variables are x, y. Isolate them left.

$$\begin{aligned} x &= 2 + 3t_1, \\ y &= 1 - 3t_1, \\ z &= t_1. \end{aligned}$$

Back-substitute. Solution found.

The solution found in the last step is called a **standard general solution**. The meaning is that all solutions of the system of equations can be found by specializing the invented symbols t_1, t_2, ... to particular numbers. Also implied is that the general solution expression satisfies the system of equations for all possible values of the symbols t_1, t_2, ...

A Fundamental Theorem

Theorem 1 (Fundamental Theorem of Toolkit Sequences)

- A general solution obtained from the last frame algorithm has the fewest possible parameters.
- The *Last Frame* is unique. Given an ordering of the variables, every last frame obtained by using the *Three Rule Toolkit* is identical.
- Any general solution having the fewest possible parameters represents each solution of the linear system by exactly one set of parameter values.

This is a *uniqueness theorem*. If a solution is written using one set of parameter values, and a second solution is written with *different parameter values*, then the two solutions have to be different. Briefly, the general solution representation is **not redundant**. Reading the proof is recommended for the mathematically inclined, after understanding the examples. See web links for the text of the proof.