Strang: Chapter 6

Section 6.1. Exercises 1, 2, 3, 4, 5, 6, 9, 16, 25, 32

Section 6.2. Exercises 4, 11, 12, 15, 16, 18, 20, 26

Section 6.3. Exercises 4, 10, 18, 19

Section 6.4. Exercises 5, 7, 11, 14, 21, 23

Section 6.5. Exercises 3, 8, 10, 23, 24, 35

Problem week13-1. Find the eigenvalues of the Markov matrix $A = \begin{pmatrix} .90 & .15 \\ .10 & .85 \end{pmatrix}$. The sum of the eigenvalues is the trace of A. What is the steady state eigenvector for the eigenvalue $\lambda_1 = 1$? See Exercise 8.3-1.

Problem week13-2. Prove that the square M^2 of a Markov matrix M is also a Markov matrix. See Exercise 8.3-9.

Problem week13-3. If A is a Markov matrix, then does $I + A + A^2 + \cdots$ add up to the *resolvent* $(A - I)^{-1}$? See Exercise 8.3-17.

Section 6.6. Exercises 3, 17, 20

Section 6.7. Exercises 1, 4, 5, 6

Some Answers

6.1. Exercises 1, 3, 6, 16, 32 have textbook answers.

6.1-2. A has $\lambda_1 = -1$ and $\lambda_2 = 5$ with eigenvectors $x_1 = (-2, 1)$ and $x_2 = (1, 1)$. The matrix $A + I$ has the same eigenvectors, with eigenvalues increased by 1 to 0 and 6. That zero eigenvalue correctly indicates that $A + I$ is singular.

6.1-4. A has $\lambda_1 = -3$ and $\lambda_2 = 2$ (check trace $= -1$ and determinant $= -6$) with $x_1 = (3, -2)$ and $x_2 = (1, 1)$. A^2 has the same eigenvectors as A, with eigenvalues $\lambda_1^2 = 9$ and $\lambda_2^2 = 4$.

6.1-5. A and B have eigenvalues 1 and 3. $A + B$ has $\lambda_1 = 3$, $\lambda_2 = 5$. Eigenvalues of $A + B$ are not equal to eigenvalues of A plus eigenvalues of B.

6.1-9. (a) Multiply by A: $A(Ax) = A(\lambda x) = \lambda Ax$ gives $A^2x = \lambda^2 x$ (b) Multiply by A^{-1}: $x = A^{-1}Ax = A^{-1}(\lambda x) = \lambda A^{-1}x$ gives $A^{-1}x = \frac{1}{\lambda}x$ (c) Add $Ix = x$: $(A + I)x = (\lambda + 1)x$.

6.1-25. With the same n eigenpairs (λ_i, x_i), then $x = c_1x_1 + \cdots + c_nx_n$ implies $Ax = c1\lambda_1x_1 + \cdots + c_n\lambda_nx_n$ and $Bx = c1\lambda_1x_1 + \cdots + c_n\lambda_nx_n$, therefore $Ax = Bx$ for all vectors x, which implies $A = B$.

6.2. Exercises 4, 12, 15, 26 have textbook answers.

6.2-11. (a) True (no zero eigenvalues) (b) False (repeated $\lambda = 2$ may have only one line of eigenvectors) (c) False (repeated λ may have a full set of eigenvectors).

6.2-16. $\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 0.2 \end{pmatrix}$, $S = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\Lambda^k \to \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $S\Lambda^kS^{-1} \to \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, which is the steady state.

6.2-18. $A^k = \frac{1}{2}\begin{pmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{pmatrix}$

6.2-20. This proof works when A is diagonalizable, $A = S\Lambda S^{-1}$:

$$\det(A) = \det(S)\det(\Lambda)\det(S^{-1}) = \det(\Lambda) = \lambda_1 \cdots \lambda_n$$

6.3. Exercise 4 has a textbook answer.

6.3-10. $A = \begin{pmatrix} 0 & 1 \\ 4 & 5 \end{pmatrix}$. $\lambda^2 - 5\lambda - 4 = 0$ is the characteristic equation of A with roots $\frac{5}{2} \pm \frac{1}{2}\sqrt{41}$. Check the characteristic equation by substitution of $y = e^{\lambda x}$ into the differential equation $y'' - 5y' - 4y = 0$.

6.3-18. Differentiate the matrix series for e^{At} as though A was a scalar to get the calculus answer $A + A^2 t + A^3 t^2/2 + \cdots$ which is exactly A times the infinite series for e^{At}.

6.3-19. $e^{Bt} = I + Bt$ (because B^2, B^3, ...are all the zero matrix). Then $e^{Bt} = \begin{pmatrix} 1 & -4t \\ 0 & 1 \end{pmatrix}$. Check $\frac{d}{dt}e^{Bt} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$ and $Be^{Bt} = B(I + Bt) = B + B^2 t = B + \text{zero matrix} = \begin{pmatrix} 0 & -4 \\ 0 & 0 \end{pmatrix}$.

6.4. Exercises 5, 11, 14, 21, 23 have textbook answers.

6.4-7. (a) $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ has eigenvalues -1 and 3. (b) Each pivot has the same signs as the λs (c) trace $= \lambda_1 + \lambda_2 = 2$, so A cannot have two negative eigenvalues.

6.5. Exercises 3, 8, 10, 24 have textbook answers.

6.5-23. $x^2/a^2 + y^2/b^2$ is $x^T Ax$ when $A = \textbf{diag}(1/a^2, 1/b^2)$. Then $\lambda_1 = 1/a^2$ and $\lambda_2 = 1/b^2$ so $a = 1/\sqrt{\lambda_1}$ and $b = 1/\sqrt{\lambda_2}$. The ellipse $9x^2 + 16y^2 = 1$ has axes with half-lengths $a = 1/3$ and $b = 1/4$. The points $(1/3, 0)$ and $(0, 1/4)$ are at the ends of the axes.

6.5-35. Put parentheses in $x^T A^T CAx$ to get $(Ax)^T C(Ax)$. Since C is assumed positive definite, this energy can drop to zero only when $Ax = 0$. Since A is assumed to have independent columns, then $Ax = 0$ only happens when $x = 0$. Thus $A^T CA$ has positive energy and it is positive definite.

Strang: My textbooks *Computational Science and Engineering* and *Introduction to Applied Mathematics* start with many examples of $A^T CA$ in a wide range of applications. I believe this is a unifying concept from linear algebra.

Problem week13-1. Eigenvalues $\lambda = 1, 0.75$; $(A - I)x = 0$ gives the steady state $x = (.6, .4)$ with $Ax = x$.

Problem week13-2. M^2 is still nonnegative; multiply M on the left by $y = [1, \ldots, 1]$ (all ones) to obtain $yM = y$. Then multiply $yM = y$ on the right by M to find $yM^2 = y$, which implies that the columns of M^2 add to 1.

Problem week13-3. No, A has an eigenvalue $\lambda = 1$ and $(I - A)^{-1}$ does not exist.

6.6. Exercise 17 has a textbook answer.

6.6-3. $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = M^{-1}AM$;

$B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

$B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

6.6-20. (a) $A = M^{-1}BM$ implies $A^2 = AA = M^{-1}B^2M$. So A^2 is similar to B^2. (b) A^2 equals $(-A)^2$ but A may not be similar to $-B$ (it could be!). (c) $\begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$ is diagonalizable to $\begin{pmatrix} 3 & 0 \\ 0 & 4 \end{pmatrix}$ because $\lambda_1 \neq \lambda_2$, so these matrices are similar. (d) $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ has only one eigenvector, so it is not diagonalizable (e) PAP^T is similar to A.

6.7. Exercises 1, 4, 5 have textbook answers.

6.7-6. $AA^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has $\sigma_1^2 = 3$ with $u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\sigma_2^2 = 1$ with $u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

$A^TA = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$ has $\sigma_1^2 = 3$ with $v_1 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, $\sigma_2^2 = 1$ with $v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $v_3 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

Then

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \mathbf{aug}(u_1, u_2)\begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\mathbf{aug}(v_1, v_2, v_3)^T$$