Strang: Chapter 3

Section 3.1. Exercises 1, 3, 11, 12, 17, 18, 22, 23

Section 3.2. Exercises 4, 10, 15, 17, 18, 19, 24, 37

Section 3.3. Exercises 1, 12, 21

Section 3.4. Exercises 1, 3, 4, 6, 13, 16, 33

Section 3.5. Exercises 2, 5, 9, 10, 11, 15, 18, 23, 26, 31, 45

Section 3.6. Exercises 1, 2, 5, 7, 11, 24

Some Answers

3.1. Exercises 1, 3, 11, 23 are in the textbook.

3.1-12. For the plane $x + y - 2z = 4$, the sum of $(4, 0, 0)$ and $(0, 4, 0)$ is not on the plane. (The key is that this plane does not go through $(0, 0, 0)$.)

3.1-17. (a) The invertible matrices do not include the zero matrix, so they are not a subspace

(b) The sum of singular matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not singular: not a subspace.

3.1-18. (a) True: The symmetric matrices do form a subspace (b) True: The matrices with $A^T = -A$ do form a subspace (c) False: The sum of two unsymmetrical matrices could be symmetric.

3.1-22. (a) Solution for every b (b) Solvable only if $b_3 = 0$. (c) Solvable only if $b_3 = b_2$.

3.2. Exercises 4, 15, 24, 37 are in the textbook.

3.2-10. (a) Impossible row 1 (b) A = invertible (c) A = all ones (d) $A = 2I$, $R = I$.

3.2-17. $A = \begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$ gives the plane $x - 3y - z = 12$; y and z are free variables. The special solutions are $(3, 1, 0)$ and $(1, 0, 1)$.

3.2-18. Fill in 12 then 4 then 1 to get the complete solution to $x - 3y - z = 12$:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 12 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = x_{\text{particular}} + x_{\text{nullspace}}$$

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3.2-19. If $LUx = 0$, multiply by L^{-1} to find $Ux = 0$. Then U and LU have the same nullspace.

3.3. Exercises 1, 21 have a textbook answer.

3.3-12. Invertible r by r submatrices use pivot rows and columns $S = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}$ and $S = [1]$ and $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3.4. Exercises 4, 6, 13, 16 have a textbook answer.

3.4-1. Row reduce the augmented matrix to upper triangular form

$$\begin{pmatrix} 2 & 4 & 6 & 4 & | & b_1 \\ 0 & 1 & 1 & 2 & | & b_2 - b_1 \\ 0 & 0 & 0 & 0 & | & b_3 + b_2 - 2b_1 \end{pmatrix}$$

Then $Ax = b$ has a solution when the last row is all zeros. This is the plane given by the equation $b_3 + b_2 - 2b_1 = 0$. The nullspace is obtained by solving $Ax = 0$, which is a step away by back-substitution. The answer is $\vec{x}_{\text{nullspace}} = c_1\vec{s}_1 + c_2\vec{s}_2$ where $\vec{s}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$, $\vec{s}_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}$. Then the complete solution is

$$c_1\vec{s}_1 + c_2\vec{s}_2 + \begin{pmatrix} b_1 \\ b_2 - b_1 \\ 0 \end{pmatrix},$$ subject to the restraint $b_3 + b_2 - 2b_1 = 0$ (b_1, b_2 unrestrained). Choosing $b_1 = 4$

and $b_2 = 3$ with $c_1 = c_2 = 0$ gives particular solution $\vec{x}_{\text{particular}} = \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \end{pmatrix}$.

3.4-3. $\vec{x}_{\text{complete}} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$. The matrix is singular but the equations are still solvable; b is in the column space. Our particular solution has free variable $y = 0$.

3.4-33. If the complete solution to $Ax = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ is $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}$ then $A = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$.

3.5. Exercises 2, 11, 15, 18 have textbook answers.

3.5-5. (a) Reduce to upper triangular $\begin{pmatrix} 1 & 2 & 3 \\ 0 & -5 & -7 \\ 0 & 0 & -18/5 \end{pmatrix}$. Invertible implies independent columns.

(b) Reduce to upper triangular $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{pmatrix}$. Then $A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, columns add to zero.

3.5-9. a) The four vectors in $\mathcal{R}^3$ are the columns of a 3 by 4 matrix A. There is a nonzero solution to $Ax = 0$ because there is at least one free variable

(b) Two vectors are dependent if $[v_1|v_2]$ has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but not v_2 is a multiple of v_1 – since v_1 might be 0.)

(c) A nontrivial combination of v_1 and 0 gives 0: $0\vec{v}_1 + 3\vec{0} = \vec{0}$.

3.5-10. The plane is the nullspace of 1 by 4 matrix $A = \begin{bmatrix} 1 & 2 & -3 & -1 \end{bmatrix}$. Three free variables give three solutions $(x, y, z, t) = (2, -1, 0, 0)$, $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$. Combinations of those special solutions give more solutions (all solutions).

3.5-23. Columns 1 and 2 are bases for the (different) column spaces of A and U; rows 1 and 2 are bases for the (equal) row spaces of A and U; $(1, -1, 1)$ is a basis for the (equal) nullspaces

3.5-26.

(a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(b) Add $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

(c) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are $3, 6, 3$.

3.5-31. (a) $y(x) = $ constant C (b) $y(x) = 3x$ (c) $y(x) = 3x + C = y_p + y_n$ solves $dy/dx = 3$.

3.5-45. If the left side of $\dim(V) + \dim(W) = \dim(V \cap W) + \dim(V+W)$ is greater than n, then $\dim(V+W) \le n$ implies $\dim(V \cap W)$ must be greater than zero. So $V \cap W$ contains nonzero vectors.

3.6. Exercises 1, 11, 24 have textbook answers.

3.6-2.

A: Row space basis = row 1 $=(1, 2, 4)$; nullspace $(-2, 1, 0)$ and $(-4, 0, 1)$; column space basis = column 1 = $(1, 2)$; left nullspace $(-2, 1)$.

B: Row space basis = both rows = $(1, 2, 4)$ and $(2, 5, 8)$; column space basis = two columns = $(1, 2)$ and $(2, 5)$; nullspace $(-4, 0, 1)$; left nullspace basis is empty because the space contains only $y = 0$.

3.6-5. $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ has those rows spanning its row space; $B = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ has the same rows spanning its nullspace and $BA^T = 0$.

3.6-7. Invertible 3 by 3 matrix A: row space basis = column space basis = $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; nullspace basis and left nullspace basis are empty. Matrix $B = \begin{bmatrix} A & A \end{bmatrix}$: row space basis $(1, 0, 0, 1, 0, 0)$, $(0, 1, 0, 0, 1, 0)$, and $(0, 0, 1, 0, 0, 1)$; column space basis $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$; nullspace basis $(-1, 0, 0, 1, 0, 0)$ and $(0, -1, 0, 0, 1, 0)$ and $(0, 0, -1, 0, 0, 1)$; left nullspace basis is empty.

3.6-28. B and C (checkers and chess) both have rank 2 if $p \ne 0$. Row 1 and row 2 are a basis for the row space of C, $B^T y = 0$ has 6 special solutions with -1 and 1 separated by a zero; $N(C^T)$ has $(-1, 0, 0, 0, 0, 0, 0, 1)$ and $(0, -1, 0, 0, 0, 0, 1, 0)$ and columns $3, 4, 5, 6$ of the identity I; $N(C)$ is a challenge.