Sample Quiz 1
2250, S2014

Problem 1, part 1.

panel 1.

$$LHS = \frac{dy}{dx}$$
$$= \frac{d}{dx}(23 - 18e^{-x})$$
$$= 0 + 18e^{-x}$$

$$RHS = -y + 23$$
$$= -(23 - 18e^{-x}) + 23$$
$$= 18e^{-x}$$

$\therefore$ LHS = RHS, DE ✓

panel 2.

$$LHS = y(0)$$
$$= (23 - 18e^{-x})\big|_{x=0}$$
$$= 23 - 18e^{0}$$
$$= 5$$
$$= RHS, \quad IC ✓$$

Problem 1, part 2.

Newton cooling is $u' = -h(u - u_1)$, $u(0) = u_0$. Changing $y \mapsto u$ and $x \mapsto t$ for the given DE + IC produces

$$\begin{cases} u' = -(u - 23), \\ u(0) = 5. \end{cases}$$

Then $h = +1$ is the cooling constant, $23 = u_1$ = ambient temperature, $5 = u_0$ = initial temperature. Then

$$\begin{cases} y(x) = u(t) = \text{apple temperature}, \\ 23 = u_1 = \text{wall thermometer temp}, \\ 5 = u_0 = \text{apple initial temp}, \\ -1 = h = \text{Newton cooling constant}, \\ x = t = \text{time}. \end{cases}$$

Sample Quiz 1

2250, S2014

Problem 2, part 1.

The tank could drain any time $t_0 < 0$ in the past, meaning there is a solution $y(x)$ such that $y(x) > 0$ for $x < t_0$ and $y(x) = 0$ for $x \geq t_0$. In short, ∞-many solutions. The model fails to determine a unique solution.

problem 2, part 2.

If $y_0 > 0$, then $f(x,y) = -0.02\sqrt{|y|}$ and $\frac{\partial f}{\partial y} = -0.01\,|y|^{-1/2}$ on box $B = \{(x,y) : |x| \leq 10, \frac{1}{2}y_0 \leq y \leq 10\}$. Picard's Theorem says there is a smaller box $B_1 = \{(x,y) : |x| \leq H, \frac{1}{2}y_0 \leq y \leq 10\}$ on which a unique edge-to-edge solution $y(x)$ exists, $y(0) = y_0$.

problem 2, part 3.

The IC is $y(0) = 19/12$ feet. Because $y > 0$, then $f(x,y) = F(x)G(y)$ with $F = -0.02$ and $G = \sqrt{y}$. Separation gives:

$$\frac{y'}{y^{1/2}} = -0.02$$

$$\int \frac{du}{u^{1/2}} = -0.02\int dx, \quad u = y(x)$$

method of quadrature ↓

$$\frac{u^{1/2}}{1/2} = -0.02x + C_1$$

$$y^{1/2} = -0.01x + C$$

$$y = (-0.01x + C)^2$$

square both sides,

$$\sqrt{\frac{19}{12}} = (0 + C)$$

$c = \sqrt{19/12}$ from 2 lines up ↑

$$y = (-0.01x + \sqrt{19/12})^2$$

Drain time is x when $y = 0$, or $x = \frac{\sqrt{19/12}}{0.01} = 125.83$

Answer checked in Wolfram Alpha and Waterloo Maple.