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Differential Equations 2280

Midterm Exam 1

Exam Date: Friday, 17 February 2017 at 12:50pm

Instructions: This in-class exam is designed for 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4.

1. (Quadrature Equations)

(a) [40%] Solve $y' = xe^{-x} + \sin^2(x)\cos(x)$.

(b) [60%] Find the position $x(t)$ from the velocity model $\dfrac{d}{dt}(e^{-t}v(t)) = 2$, $v(0) = 5$ and the position model $\dfrac{dx}{dt} = v(t)$, $x(0) = 2$.

Solution to Problem 1.

(a) Answer $y = -xe^{-x} - e^{-x} + \dfrac{1}{3}\sin^3(x) + c$. Treat the problem as a quadrature problem $y' = F(x)$, then $y = \int F(x)dx$. Integration details:

$$
\begin{aligned}
\int F(x)dx &= \int xe^{-x}\,dx + \int \sin^2(x)\cos(x)\,dx \\
&= I_1 + I_2. \\
I_1 &= \int xe^{-x}\,dx \\
&= -xe^{-x} - \int e^{-x}\,dx, \quad \text{parts} \quad u = x, dv = e^{-x}dx, \\
&= -xe^{-x} - e^{-x} + c_1 \\
I_2 &= \int \sin^2(x)\cos(x)\,dx \\
&= \int u^2\,du, \quad u = \sin(x), du = \cos(x)dx, \\
&= u^3/3 + c_2 \\
&= \tfrac{1}{3}\sin^3(x) + c_2
\end{aligned}
$$

(b) Velocity $v(t) = 2t\,e^t + 5\,e^t$ by quadrature. Integrate $x'(t) = 2t\,e^t + 5e^t$ with $x(0) = 2$ to obtain position $x(t) = (2t+3)e^t - 1$. The integral of te^t is found using integration by parts. See Exercise 1.2-10 in Edwards-Penney and the solution to (a) above.

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2. (Solve a Separable Equation)
The differential equation $y' = f(x,y)$ is defined to be **separable** provided $f(x,y) = F(x)G(y)$ for some functions F and G.

(a) [30%] The equation $y' + x(y+3) = ye^x + 3x$ is separable. Provide formulas for the functions F and G.

(b) [70%] Find a non-equilibrium solution in implicit form for the separable equation

$$(10)y' = \left(\frac{1}{1+x^2} + \ln|x|\right)(y^2 - 3y + 2)$$

To save time, **do not solve** for y explicitly and **do not solve** for equilibrium solutions.

Solution to Problem 2.

(a) The equation is $y' = ye^x - xy = (e^x - x)y$. Then $F(x) = e^x - x$, $G(y) = y$.

(b) The solution by separation of variables identifies the separated equation $y' = F(x)G(y)$ using definitions

$$F(x) = \frac{1}{1+x^2} + \ln|x|, \quad G(y) = \frac{y^2 - 3y + 2}{10}.$$

The integral of F is from standard formulas and/or integration by parts.

$$\begin{aligned}
\int F dx &= \int \frac{1}{1+x^2} + \ln|x|\, dx \\
&= I_1 + I_2.
\end{aligned}$$

$$\begin{aligned}
I_1 &= \int \frac{1}{1+x^2}\, dx \\
&= \arctan(x) + c_1, \\
I_2 &= \int \ln|x|\, dx \\
&= \int u\, dv, \quad \text{parts} \quad u = \ln|x|, dv = dx, \\
&= uv - \int v\, du, \quad v = x,\ du = dx/x, \\
&= x\ln|x| - \int 1\, dx, \\
&= x\ln|x| - x + c_2.
\end{aligned}$$

Then $\displaystyle\int F(x)dx = \arctan(x) + x\ln|x| - x + c_3$.

The integral of $1/G(y)$ requires partial fractions. The details:

$$\begin{aligned}
\int \frac{dx}{G(y(x))} &= \int \frac{10}{u^2 - 3u + 2}\, du, \quad u = y(x), du = y'(x)dx, \\
&= \int \frac{10}{(u-2)(u-1)}\, du \\
&= \int \frac{A}{u-2} + \frac{B}{u-1}\, du, \quad A, B \quad \text{determined later}, \\
&= A\ln|u-2| + B\ln|u-1| + c_4
\end{aligned}$$

The partial fraction problem

$$\frac{10}{(u-2)(u-1)} = \frac{A}{u-2} + \frac{B}{u-1}$$

can be solved in a variety of ways, with answer $A = 10$ and $B = -10$. The final implicit solution is obtained from $\int \frac{dx}{G(y(x))} = \int F(x)dx$, which gives the equation

$$10\ln|y-2| - 10\ln|y-1| = \arctan(x) + x\ln|x| - x + c.$$

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3. (Linear Equations)

(a) [60%] Solve the linear model $2x'(t) = -32 + \dfrac{10}{3t+2}x(t)$, $x(0) = 16$. Show all integrating factor steps.

(b) [20%] Solve $\dfrac{dy}{dx} + (\sin(x))y = 0$ using the homogeneous linear equation shortcut.

(c) [20%] Solve $5\dfrac{dy}{dx} = 21y + 7$ using the superposition principle $y = y_h + y_p$ shortcut. Expected are answers for y_h and y_p.

Solution to Problem 3.

(a) The answer is $v(t) = 16 + 24t$. The details:

$$\begin{aligned}
&v'(t) = -16 + \frac{5}{3t+2}v(t), \\
&v'(t) + \frac{-5}{3t+2}v(t) = -16, \quad \text{standard form } v' + p(t)v = q(t) \\
&p(t) = \frac{-5}{3t+2}, \\
&W = e^{\int p\,dt}, \quad \text{integrating factor} \\
&W = e^u, \quad u = \textstyle\int p\,dt = -\frac{5}{3}\ln|3t+2| = \ln\left(|3t+2|^{-5/3}\right) \\
&W = (3t+2)^{-5/3}, \quad \text{Final choice for } W.
\end{aligned}$$

Then replace the left side of $v' + pv = q$ by $(vW)'/W$ to obtain

$$\begin{aligned}
&v'(t) + \frac{-5}{3t+2}v(t) = -16, \quad \text{standard form } v' + p(t)v = q(t) \\
&\frac{(vW)'}{W} = -32, \quad \text{Replace left side by quotient } (vW)'/W \\
&(vW)' = -16W, \quad \text{cross-multiply} \\
&vW = -16\textstyle\int W\,dt, \quad \text{quadrature step.}
\end{aligned}$$

The evaluation of the integral is from the power rule:

$$\int -16W\,dt = -16\int(3t+2)^{-5/3}dt = -32\frac{(3t+2)^{-2/3}}{(-2/3)(3)} + c.$$

Division by $W = (3t+2)^{-5/3}$ then gives the general solution

$$v(t) = \frac{c}{W} - \frac{16}{-2}(3t+2)^{-2/3}(3t+2)^{5/3}.$$

Constant c evaluates to $c = 0$ because of initial condition $v(0) = 16$. Then

$$v(t) = \frac{16}{-2}(3t+2)^{-2/3}(3t+2)^{5/3} = 8(3t+2)^{-\frac{2}{3}+\frac{5}{3}} = 8(3t+2).$$

(b) The answer is y = constant divided by the integrating factor: $y = \dfrac{c}{W}$. Because $W = e^u$ where $u = \int \sin(x)dx = -\cos x$, then $y = ce^{\cos x}$.

(c) The equilibrium solution (a constant solution) is $y_p = -\frac{7}{21}$. The homogeneous solution is $y_h = ce^{21x/5}$ = constant divided by the integrating factor. Then $y = y_p + y_h = -\frac{1}{3} + c\,e^{21x/5}$.

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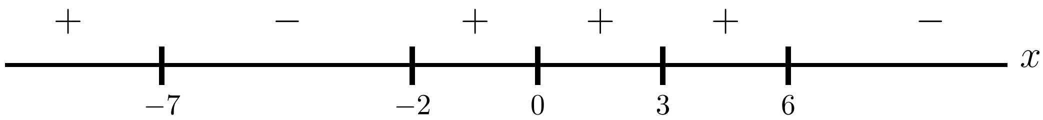
4. (Stability)

Assume an autonomous equation $x'(t) = f(x(t))$. Draw a phase portrait with at least 12 threaded curves, using the phase line diagram given below. Add these labels as appropriate: funnel, spout, node [neither spout nor funnel], stable, unstable.



Solution to Problem 4.
The graphic is drawn using increasing and decreasing curves, which may or may not be depicted with turning points. The rules:

1. A curve drawn between equilibria is increasing if the sign is PLUS.
2. A curve drawn between equilibria is decreasing if the sign is MINUS.
3. Label: FUNNEL, STABLE
   The signs left to right are PLUS MINUS crossing the equilibrium point.
4. Label: SPOUT, UNSTABLE
   The signs left to right are MINUS PLUS crossing the equilibrium point.
5. Label: NODE, UNSTABLE
   The signs left to right are PLUS PLUS crossing the equilibrium point, or
   The signs left to right are MINUS MINUS crossing the equilibrium point.

The answer:

$x = -7$: FUNNEL, STABLE
$x = -2$: SPOUT, UNSTABLE
$x = 0$: NODE, UNSTABLE
$x = 3$: NODE, UNSTABLE
$x = 6$: FUNNEL, STABLE

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5. (ch3)

Using Euler's theorem on Euler solution atoms and the characteristic equation for higher order constant-coefficient differential equations, solve (a), (b), (c).

(a) [40%] Find a constant coefficient differential equation $ay''' + by'' + cy' + dy = 0$ which has a particular solution $-5e^{-x} + xe^{-x} + 10$.

(b) [30%] Given characteristic equation $r^2(r+2)(r^3-2r)(r^2+2r+17) = 0$, solve the differential equation.

(c) [20%] Given $mx''(t) + cx'(t) + kx(t) = 0$, which represents an unforced damped spring-mass system. Assume $m = 4$, $c = 4$, $k = 65$. Classify the equation as over-damped, critically damped or under-damped. Illustrate in a spring-mass-dashpot drawing the assignment of physical constants m, c, k and the initial conditions $x(0) = 1$, $x'(0) = 0$.

(d) [10%] Given $mx''(t) + cx'(t) + kx(t) = 0$, which represents an unforced damped spring-mass system. Assume $m = 4$, $c = 4$, $k = 65$. Illustrate in a spring-mass-dashpot drawing the assignment of physical constants m, c, k and coordinates for $x(t)$. Explain the physical meaning of the initial conditions $x(0) = 1$, $x'(0) = 0$.

Solution to Problem 5.

5(a)

A solution of a constant-coefficient linear homogeneous differential equation is a linear combinations of Euler solution atoms. The given particular solution is a linear combination of Euler atoms e^{-x}, $x\,e^{-x}$, 1. Because $1 = e^{0x}$, then one root of the characteristic equation is $r = 0$. Due to Euler's multiplicity theorem, the Euler atoms e^{-x}, $x\,e^{-x}$ account for a double root $r = -1, -1$. Then the characterisitic equation has factors $(r+1)$, $(r+1)$, r [College Algebra Root-Factor Theorem applied]. The characteristic equation is then

$$(r+1)(r+1)r = 0$$

which is $r^3 + 2r^2 + r = 0$. Solving backwards gives the differential equation $y''' + 2y'' + y' = 0$.

The answer is the differential equation $y''' + 2y'' + y' = 0$.

5(b)

The characteristic equation factors into $r^3(r+2)(r^2-2)(r^2+2r+17) = 0$ with roots $r = 0, 0, 0; -2; \sqrt{2}; -\sqrt{2}; -1 \pm 4i$. Then y is a linear combination of the Euler solution atoms, which are

$$1, x, x^2, e^{-2x}, e^{x\sqrt{2}}, e^{-x\sqrt{2}}, e^{-x}\cos(4x), e^{-x}\sin(4x)$$

5(c)

Use $4r^2 + 4r + 65 = 0$ and the quadratic formula to obtain roots $r = -1/2 + 4i, -1/2 - 4i$ and Euler solution atoms $e^{-x/2}\cos 4t, e^{-x/2}\sin 4t$. Then y is a linear combination of these two solution atoms, and it oscillates, therefore the classification is **under-damped**.

5(d)

The illustration shows a spring, a dashpot and a mass with labels k, c, m. Initial conditions mean mass elongation $x = 1$, at rest.

A **dashpot** is represented as a cylinder and piston with rod, the rod attached to the mass. Variable x is positive in the down direction and negative in the up direction. The equilibrium position is $x = 0$.

The physical meaning of $x(0) = 1$ and $x'(0) = 0$ is the mass is pulled downward one unit from equilibrium $(x = 0)$ and released.