11.4 Matrix Exponential

The problem

$$\mathbf{x}'(t) = A\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

has a unique solution, according to the Picard-Lindelöf theorem. Solve the problem n times, when $\mathbf{x}_0$ equals a column of the identity matrix, and write $\mathbf{w}_1(t), \ldots, \mathbf{w}_n(t)$ for the n solutions so obtained. Define the **matrix exponential** by packaging these n solutions into a matrix:

$$e^{At} \equiv \mathbf{aug}(\mathbf{w}_1(t), \ldots, \mathbf{w}_n(t)).$$

By construction, any possible solution of $\mathbf{x}' = A\mathbf{x}$ can be uniquely expressed in terms of the matrix exponential e^{At} by the formula

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0).$$

Matrix Exponential Identities

Announced here and proved below are various formulas and identities for the matrix exponential e^{At}:

$\left(e^{At}\right)' = Ae^{At}$	Columns satisfy $\mathbf{x}' = A\mathbf{x}$.
$e^{\mathbf{0}} = I$	Where $\mathbf{0}$ is the zero matrix.
$Be^{At} = e^{At}B$	If $AB = BA$.
$e^{At}e^{Bt} = e^{(A+B)t}$	If $AB = BA$.
$e^{At}e^{As} = e^{A(t+s)}$	At and As commute.
$\left(e^{At}\right)^{-1} = e^{-At}$	Equivalently, $e^{At}e^{-At} = I$.
$e^{At} = r_1(t)P_1 + \cdots + r_n(t)P_n$	Putzer's spectral formula — see page 776.
$e^{At} = e^{\lambda_1 t}I + \dfrac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I)$	A is 2×2, $\lambda_1 \neq \lambda_2$ real.
$e^{At} = e^{\lambda_1 t}I + te^{\lambda_1 t}(A - \lambda_1 I)$	A is 2×2, $\lambda_1 = \lambda_2$ real.
$e^{At} = e^{at}\cos bt\, I + \dfrac{e^{at}\sin bt}{b}(A - aI)$	A is 2×2, $\lambda_1 = \overline{\lambda}_2 = a+ib$, $b > 0$.
$e^{At} = \displaystyle\sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$	Picard series. See page 777.
$e^{At} = P^{-1}e^{Jt}P$	Jordan form $J = PAP^{-1}$.

Putzer's Spectral Formula

The spectral formula of Putzer applies to a system $\mathbf{x}' = A\mathbf{x}$ to find its general solution. The method uses matrices $P_1, \ldots, P_n$ constructed from A and the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A, matrix multiplication, and the solution $\mathbf{r}(t)$ of the first order $n \times n$ initial value problem

$$\mathbf{r}'(t) = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 & 0 \\ & & & \vdots & & \\ 0 & 0 & 0 & \cdots & 1 & \lambda_n \end{pmatrix} \mathbf{r}(t), \quad \mathbf{r}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The system is solved by first order scalar methods and back-substitution. We will derive the formula separately for the 2×2 case (the one used most often) and the $n \times n$ case.

Spectral Formula 2×2

The general solution of the 2×2 system $\mathbf{x}' = A\mathbf{x}$ is given by the formula

$$\mathbf{x}(t) = \left(r_1(t)P_1 + r_2(t)P_2\right)\mathbf{x}(0),$$

where r_1, r_2, P_1, P_2 are defined as follows.

The eigenvalues $r = \lambda_1, \lambda_2$ are the two roots of the quadratic equation

$$\det(A - rI) = 0.$$

Define 2×2 matrices P_1, P_2 by the formulas

$$P_1 = I, \quad P_2 = A - \lambda_1 I.$$

The functions $r_1(t)$, $r_2(t)$ are defined by the differential system

$$\begin{aligned} r_1' &= \lambda_1 r_1, & r_1(0) &= 1, \\ r_2' &= \lambda_2 r_2 + r_1, & r_2(0) &= 0. \end{aligned}$$

Proof: The Cayley-Hamilton formula $(A - \lambda_1 I)(A - \lambda_2 I) = \mathbf{0}$ is valid for any 2×2 matrix A and the two roots $r = \lambda_1, \lambda_2$ of the determinant equality $\det(A - rI) = 0$. The Cayley-Hamilton formula is the same as $(A - \lambda_2)P_2 = \mathbf{0}$, which implies the identity $AP_2 = \lambda_2 P_2$. Compute as follows.

$$\begin{aligned} \mathbf{x}'(t) &= \left(r_1'(t)P_1 + r_2'(t)P_2\right)\mathbf{x}(0) \\ &= \left(\lambda_1 r_1(t)P_1 + r_1(t)P_2 + \lambda_2 r_2(t)P_2\right)\mathbf{x}(0) \\ &= \left(r_1(t)A + \lambda_2 r_2(t)P_2\right)\mathbf{x}(0) \\ &= \left(r_1(t)A + r_2(t)AP_2\right)\mathbf{x}(0) \\ &= A\left(r_1(t)I + r_2(t)P_2\right)\mathbf{x}(0) \\ &= A\mathbf{x}(t). \end{aligned}$$

This proves that $\mathbf{x}(t)$ is a solution. Because $\Phi(t) \equiv r_1(t)P_1 + r_2(t)P_2$ satisfies $\Phi(0) = I$, then any possible solution of $\mathbf{x}' = A\mathbf{x}$ can be represented by the given formula. The proof is complete.

Real Distinct Eigenvalues. Suppose A is 2×2 having real distinct eigenvalues λ_1, λ_2 and $\mathbf{x}(0)$ is real. Then

$$r_1 = e^{\lambda_1 t}, \quad r_2 = \frac{e^{\lambda_1 t} - e^{\lambda_2 T}}{\lambda_1 - \lambda_2}$$

and

$$\mathbf{x}(t) = \left(e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I) \right) \mathbf{x}(0).$$

The matrix exponential formula for real distinct eigenvalues:

$$e^{At} = e^{\lambda_1 t} I + \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}(A - \lambda_1 I).$$

Real Equal Eigenvalues. Suppose A is 2×2 having real equal eigenvalues $\lambda_1 = \lambda_2$ and $\mathbf{x}(0)$ is real. Then $r_1 = e^{\lambda_1 t}$, $r_2 = te^{\lambda_1 t}$ and

$$\mathbf{x}(t) = \left(e^{\lambda_1 t} I + te^{\lambda_1 t}(A - \lambda_1 I) \right) \mathbf{x}(0).$$

The matrix exponential formula for real equal eigenvalues:

$$e^{At} = e^{\lambda_1 t} I + te^{\lambda_1 t}(A - \lambda_1 I).$$

Complex Eigenvalues. Suppose A is 2×2 having complex eigenvalues $\lambda_1 = a + bi$ with $b > 0$ and $\lambda_2 = a - bi$. If $\mathbf{x}(0)$ is real, then a real solution is obtained by taking the real part of the spectral formula. This formula is formally identical to the case of real distinct eigenvalues. Then

$$\begin{aligned}
\mathcal{R}\mathrm{e}(\mathbf{x}(t)) &= \left(\mathcal{R}\mathrm{e}(r_1(t))I + \mathcal{R}\mathrm{e}(r_2(t)(A - \lambda_1 I))\right)\mathbf{x}(0) \\
&= \left(\mathcal{R}\mathrm{e}(e^{(a+ib)t})I + \mathcal{R}\mathrm{e}(e^{at}\frac{\sin bt}{b}(A - (a+ib)I)) \right) \mathbf{x}(0) \\
&= \left(e^{at}\cos bt\, I + e^{at}\frac{\sin bt}{b}(A - aI)) \right) \mathbf{x}(0)
\end{aligned}$$

The matrix exponential formula for complex conjugate eigenvalues:

$$e^{At} = e^{at}\left(\cos bt\, I + \frac{\sin bt}{b}(A - aI)) \right).$$

How to Remember Putzer's 2×2 Formula. The expressions

$$\begin{aligned}
&e^{At} = r_1(t)I + r_2(t)(A - \lambda_1 I), \\
&r_1(t) = e^{\lambda_1 t}, \quad r_2(t) = \frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}
\end{aligned} \tag{1}$$

are enough to generate all three formulas. Fraction r_2 is the $d/d\lambda$-Newton quotient for r_1. It has limit $te^{\lambda_1 t}$ as $\lambda_2 \to \lambda_1$, therefore the formula includes the case $\lambda_1 = \lambda_2$ by limiting. If $\lambda_1 = \overline{\lambda}_2 = a + ib$ with $b > 0$, then the fraction r_2 is already real, because it has for $z = e^{\lambda_1 t}$ and $w = \lambda_1$ the form

$$r_2(t) = \frac{z - \overline{z}}{w - \overline{w}} = \frac{\sin bt}{b}.$$

Taking real parts of expression (1) gives the complex case formula.

Spectral Formula $n \times n$

The general solution of $\mathbf{x}' = A\mathbf{x}$ is given by the formula

$$\mathbf{x}(t) = (r_1(t)P_1 + r_2(t)P_2 + \cdots + r_n(t)P_n)\,\mathbf{x}(0),$$

where r_1, r_2, $\ldots$, r_n, P_1, P_2, $\ldots$, P_n are defined as follows.

The eigenvalues $r = \lambda_1, \ldots, \lambda_n$ are the roots of the polynomial equation

$$\det(A - rI) = 0.$$

Define $n \times n$ matrices P_1, $\ldots$, P_n by the formulas

$$P_1 = I, \quad P_k = P_{k-1}(A - \lambda_{k-1}I) = \Pi_{j=1}^{k-1}(A - \lambda_j I), \quad k = 2, \ldots, n.$$

The functions $r_1(t)$, $\ldots$, $r_n(t)$ are defined by the differential system

$$\begin{array}{rcll} r_1' &=& \lambda_1 r_1, & r_1(0) = 1, \\ r_2' &=& \lambda_2 r_2 + r_1, & r_2(0) = 0, \\ &\vdots& & \\ r_n' &=& \lambda_n r_n + r_{n-1}, & r_n(0) = 0. \end{array}$$

Proof: The Cayley-Hamilton formula $(A - \lambda_1 I)\cdots(A - \lambda_n I) = \mathbf{0}$ is valid for any $n \times n$ matrix A and the n roots $r = \lambda_1, \ldots, \lambda_n$ of the determinant equality $\det(A - rI) = 0$. Two facts will be used: (1) The Cayley-Hamilton formula implies $AP_n = \lambda_n P_n$; (2) The definition of P_k implies $\lambda_k P_k + P_{k+1} = AP_k$ for $1 \le k \le n - 1$. Compute as follows.

$$\boxed{\mathbf{1}} \quad \mathbf{x}'(t) = (r_1'(t)P_1 + \cdots + r_n'(t)P_n)\,\mathbf{x}(0)$$

$$\boxed{\mathbf{2}} \quad = \left(\sum_{k=1}^{n} \lambda_k r_k(t) P_k + \sum_{k=2}^{n} r_{k-1} P_k\right)\mathbf{x}(0)$$

$$\boxed{\mathbf{3}} \quad = \left(\sum_{k=1}^{n-1} \lambda_k r_k(t) P_k + r_n(t)\lambda_n P_n + \sum_{k=1}^{n-1} r_k P_{k+1}\right)\mathbf{x}(0)$$

$$\boxed{\mathbf{4}} \quad = \left(\sum_{k=1}^{n-1} r_k(t)(\lambda_k P_k + P_{k+1}) + r_n(t)\lambda_n P_n\right)\mathbf{x}(0)$$

$$\boxed{\mathbf{5}} \quad = \left(\sum_{k=1}^{n-1} r_k(t) A P_k + r_n(t) A P_n\right)\mathbf{x}(0)$$

$$\boxed{\mathbf{6}} \quad = A\left(\sum_{k=1}^{n} r_k(t) P_k\right)\mathbf{x}(0)$$

$$\boxed{\mathbf{7}} \quad = A\mathbf{x}(t).$$

Details: $\boxed{1}$ Differentiate the formula for $\mathbf{x}(t)$. $\boxed{2}$ Use the differential equations for $r_1, \ldots, r_n$. $\boxed{3}$ Split off the last term from the first sum, then re-index the last sum. $\boxed{4}$ Combine the two sums. $\boxed{5}$ Use the recursion for P_k and the Cayley-Hamilton formula $(A - \lambda_n I)P_n = \mathbf{0}$. $\boxed{6}$ Factor out A on the left. $\boxed{7}$ Apply the definition of $\mathbf{x}(t)$.

This proves that $\mathbf{x}(t)$ is a solution. Because $\Phi(t) \equiv \sum_{k=1}^n r_k(t)P_k$ satisfies $\Phi(0) = I$, then any possible solution of $\mathbf{x}' = A\mathbf{x}$ can be so represented. The proof is complete.

Proofs of Matrix Exponential Properties

Verify $\left(e^{At}\right)' = Ae^{At}$. Let $\mathbf{x}_0$ denote a column of the identity matrix. Define $\mathbf{x}(t) = e^{At}\mathbf{x}_0$. Then

$$\begin{aligned} \left(e^{At}\right)'\mathbf{x}_0 &= \mathbf{x}'(t) \\ &= A\mathbf{x}(t) \\ &= Ae^{At}\mathbf{x}_0. \end{aligned}$$

Because this identity holds for all columns of the identity matrix, then $(e^{At})'$ and Ae^{At} have identical columns, hence we have proved the identity $\left(e^{At}\right)' = Ae^{At}$.

Verify $AB = BA$ implies $Be^{At} = e^{At}B$. Define $\mathbf{w}_1(t) = e^{At}B\mathbf{w}_0$ and $\mathbf{w}_2(t) = Be^{At}\mathbf{w}_0$. Calculate $\mathbf{w}_1'(t) = A\mathbf{w}_1(t)$ and $\mathbf{w}_2'(t) = BAe^{At}\mathbf{w}_0 = ABe^{At}\mathbf{w}_0 = A\mathbf{w}_2(t)$, due to $BA = AB$. Because $\mathbf{w}_1(0) = \mathbf{w}_2(0) = \mathbf{w}_0$, then the uniqueness assertion of the Picard-Lindelöf theorem implies that $\mathbf{w}_1(t) = \mathbf{w}_2(t)$. Because $\mathbf{w}_0$ is any vector, then $e^{At}B = Be^{At}$. The proof is complete.

Verify $e^{At}e^{Bt} = e^{(A+B)t}$. Let $\mathbf{x}_0$ be a column of the identity matrix. Define $\mathbf{x}(t) = e^{At}e^{Bt}\mathbf{x}_0$ and $\mathbf{y}(t) = e^{(A+B)t}\mathbf{x}_0$. We must show that $\mathbf{x}(t) = \mathbf{y}(t)$ for all t. Define $\mathbf{u}(t) = e^{Bt}\mathbf{x}_0$. We will apply the result $e^{At}B = Be^{At}$, valid for $BA = AB$. The details:

$$\begin{aligned} \mathbf{x}'(t) &= \left(e^{At}\mathbf{u}(t)\right)' \\ &= Ae^{At}\mathbf{u}(t) + e^{At}\mathbf{u}'(t) \\ &= A\mathbf{x}(t) + e^{At}B\mathbf{u}(t) \\ &= A\mathbf{x}(t) + Be^{At}\mathbf{u}(t) \\ &= (A+B)\mathbf{x}(t). \end{aligned}$$

We also know that $\mathbf{y}'(t) = (A+B)\mathbf{y}(t)$ and since $\mathbf{x}(0) = \mathbf{y}(0) = \mathbf{x}_0$, then the Picard-Lindelöf theorem implies that $\mathbf{x}(t) = \mathbf{y}(t)$ for all t. This completes the proof.

Verify $e^{At}e^{As} = e^{A(t+s)}$. Let t be a variable and consider s fixed. Define $\mathbf{x}(t) = e^{At}e^{As}\mathbf{x}_0$ and $\mathbf{y}(t) = e^{A(t+s)}\mathbf{x}_0$. Then $\mathbf{x}(0) = \mathbf{y}(0)$ and both satisfy the differential equation $\mathbf{u}'(t) = A\mathbf{u}(t)$. By the uniqueness in the Picard-Lindelöf theorem, $\mathbf{x}(t) = \mathbf{y}(t)$, which implies $e^{At}e^{As} = e^{A(t+s)}$. The proof is complete.

Verify $e^{At} = \displaystyle\sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$. The idea of the proof is to apply Picard iteration. By definition, the columns of e^{At} are vector solutions $\mathbf{w}_1(t), \ldots, \mathbf{w}_n(t)$ whose values at $t = 0$ are the corresponding columns of the $n \times n$ identity matrix. According to the theory of Picard iterates, a particular iterate is defined by

$$\mathbf{y}_{n+1}(t) = \mathbf{y}_0 + \int_0^t A\mathbf{y}_n(r)dr, \quad n \geq 0.$$

The vector $\mathbf{y}_0$ equals some column of the identity matrix. The Picard iterates can be found explicitly, as follows.

$$\begin{aligned}
\mathbf{y}_1(t) &= \mathbf{y}_0 + \textstyle\int_0^t A\mathbf{y}_0 dr \\
&= (I + At)\,\mathbf{y}_0, \\
\mathbf{y}_2(t) &= \mathbf{y}_0 + \textstyle\int_0^t A\mathbf{y}_1(r)dr \\
&= \mathbf{y}_0 + \textstyle\int_0^t A\,(I + At)\,\mathbf{y}_0 dr \\
&= \left(I + At + A^2t^2/2\right)\mathbf{y}_0, \\
&\vdots \\
\mathbf{y}_n(t) &= \left(I + At + A^2\tfrac{t^2}{2} + \cdots + A^n\tfrac{t^n}{n!}\right)\mathbf{y}_0.
\end{aligned}$$

The Picard-Lindelöf theorem implies that for $\mathbf{y}_0$ = column k of the identity matrix,

$$\lim_{n\to\infty} \mathbf{y}_n(t) = \mathbf{w}_k(t).$$

This being valid for each index k, then the columns of the matrix sum

$$\sum_{m=0}^{N} A^m \frac{t^m}{m!}$$

converge as $N \to \infty$ to $\mathbf{w}_1(t), \ldots, \mathbf{w}_n(t)$. This implies the matrix identity

$$e^{At} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}.$$

The proof is complete.

Computing e^{At}

Theorem 13 (Computing e^{Jt} for J Triangular)
If J is an upper triangular matrix, then a column $\mathbf{u}(t)$ of e^{Jt} can be computed by solving the system $\mathbf{u}'(t) = J\mathbf{u}(t)$, $\mathbf{u}(0) = \mathbf{v}$, where $\mathbf{v}$ is the corresponding column of the identity matrix. This problem can always be solved by first-order scalar methods of growth-decay theory and the integrating factor method.

Theorem 14 (Exponential of a Diagonal Matrix)
For real or complex constants $\lambda_1, \ldots, \lambda_n$,

$$e^{\mathbf{diag}(\lambda_1,\ldots,\lambda_n)t} = \mathbf{diag}\left(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right).$$

Theorem 15 (Block Diagonal Matrix)
If $A = \textbf{diag}(B_1, \ldots, B_k)$ and each of $B_1, \ldots, B_k$ is a square matrix, then

$$e^{At} = \textbf{diag}\left(e^{B_1 t}, \ldots, e^{B_k t}\right).$$

Theorem 16 (Complex Exponential)
Given real a, b, then

$$e^{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} t} = e^{at} \begin{pmatrix} \cos bt & \sin bt \\ -\sin bt & \cos bt \end{pmatrix}.$$

Exercises 11.4

Matrix Exponential.

1. **(Picard)** Let A be real 2×2. Write out the two initial value problems which define the columns $\mathbf{w}_1(t)$, $\mathbf{w}_2(t)$ of e^{At}.

2. **(Picard)** Let A be real 3×3. Write out the three initial value problems which define the columns $\mathbf{w}_1(t)$, $\mathbf{w}_2(t)$, $\mathbf{w}_3(t)$ of e^{At}.

3. **(Definition)** Let A be real 2×2. Show that the solution $\mathbf{x}(t) = e^{At}\mathbf{u}_0$ satisfies $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{x}(0) = \mathbf{u}_0$.

4. **Definition** Let A be real $n \times n$. Show that the solution $\mathbf{x}(t) = e^{At}\mathbf{x}(0)$ satisfies $\mathbf{x}' = A\mathbf{x}$.

Matrix Exponential 2×2. Find e^{At} using the formula $e^{At} = \textbf{aug}(\mathbf{w}_1, \mathbf{w}_2)$ and the corresponding systems $\mathbf{w}_1' = A\mathbf{w}_1$, $\mathbf{w}_1(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{w}_2' = A\mathbf{w}_2$, $\mathbf{w}_2(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In these exercises A is triangular so that first-order methods can solve the systems.

5. $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

6. $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$.

7. $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$.

8. $A = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.

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