Problem 1. (Chapter 1: 100 points) Solve Linear Algebraic Equations.

Solve the system $A\vec{u} = \vec{b}$ defined by

$$A \quad \begin{cases} 2x_1 + x_2 + 8x_3 + x_4 + 2x_5 &= 4 \\ x_1 + 3x_2 + 4x_3 + x_4 + x_5 &= 2 \\ 2x_1 + 2x_2 + 8x_3 + x_4 + x_5 &= 4 \end{cases} \qquad \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}.$$

Expected: (a) Augmented matrix and Toolkit steps to the reduced row echelon form (RREF). (b) Translation of RREF to scalar equations. Identify lead and free variables. (c) Scalar general solution for x_1 to x_5. (d) Write vector formulas for the homogeneous solution $\vec{u}_h$, a particular solution $\vec{u}_p$ and the general solution $\vec{u} = \vec{u}_h + \vec{u}_p$.

a)
$$\left[\begin{array}{ccccc|c} 2 & 1 & 8 & 1 & 2 & 4 \\ 1 & 3 & 4 & 1 & 1 & 2 \\ 2 & 2 & 8 & 1 & 1 & 4 \end{array}\right] \begin{array}{l} \text{combo}(2,1,-2) \\ \text{swap}(1,2) \\ \text{combo}(2,3,-2) \end{array} \begin{bmatrix} 1 & 3 & 4 & 1 & 1 & 2 \\ 0 & -5 & 0 & -1 & 0 & 0 \\ 0 & -4 & 0 & -1 & -1 & 0 \end{bmatrix} \begin{array}{l} \text{mult}(2,\tfrac{-1}{5}) \\ \text{combo}(2,3,-\tfrac{4}{5}) \end{array} \begin{bmatrix} 1 & 3 & 4 & 1 & 1 & 2 \\ 0 & 1 & 0 & \tfrac{1}{5} & 0 & 0 \\ 0 & 0 & 0 & -\tfrac{1}{5} & -1 & 0 \end{bmatrix} \begin{array}{l} \text{combo}(3,1,5) \\ \text{combo}(3,2,1) \\ \text{mult}(3,-5) \end{array}$$

$$\begin{bmatrix} 1 & 3 & 4 & 0 & -4 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{bmatrix} \text{combo}(2,1,-3) \begin{bmatrix} 1 & 0 & 4 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 5 & 0 \end{bmatrix}$$

b)
$x_1 + 4x_3 - x_5 = 2$
$x_2 - x_5 = 0$
$x_4 + 5x_5 = 0$
x_3, x_5 free variables
x_1, x_2, x_4 lead variables

c)
$x_1 = 2 - 4x_3 + x_5$
$x_2 = x_5$
$x_3 = x_3$
$x_4 = -5x_5$
$x_5 = x_5$

d)
$$\vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 - 4x_3 + x_5 \\ x_5 \\ x_3 \\ -5x_5 \\ x_5 \end{bmatrix} = \boxed{\underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\vec{u}_p} + \underbrace{x_3 \begin{bmatrix} -4 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 1 \\ 0 \\ -5 \\ 1 \end{bmatrix}}_{\vec{u}_h} = \vec{u}}$$

Problem 2. (Chapters 2 and 3: 100 points) Inverse, Determinant, Cramer's Rule.

A **(a)** [40%] Find the inverse of the matrix $A = \begin{pmatrix} 1 & 4 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

A **(b)** [30%] Let A be defined as in part (a). Compute the determinant of $((A + A^T)^{-1})^T$.

A **(c)** [30%] Let P, Q, R denote undisclosed real numbers. Define matrix B and vectors $\vec{\mathbf{x}}$ and $\vec{\mathbf{c}}$ by the equations

$$B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & 2 \end{pmatrix}, \quad \vec{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{\mathbf{c}} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

Find the value of x_3 by Cramer's Rule in the system $B\vec{\mathbf{x}} = \vec{\mathbf{c}}$.

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a) $\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 0 & 0 \\ 1 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ combo(1,2,-1) $\left[\begin{array}{ccc|ccc} 1 & 4 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ combo(2,1,-4) $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -4 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$

$$\boxed{A^{-1} = \begin{bmatrix} 5 & -4 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

b) $((A+A^T)^{-1})^T = ((A+A^T)^T)^{-1} = (A^T+A)^{-1} = \left(\begin{bmatrix} 1 & 1 & 0 \\ 4 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 0 \\ 1 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 5 & 0 \\ 5 & 10 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{-1}$

$\left[\begin{array}{ccc|ccc} 2 & 5 & 0 & 1 & 0 & 0 \\ 5 & 10 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$ mult(1,$\frac{1}{2}$) combo(1,2,$-\frac{5}{2}$) mult(3,$\frac{1}{2}$) $\left[\begin{array}{ccc|ccc} 1 & \frac{5}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & -\frac{5}{2} & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$ combo(2,1,1) mult(2,$-\frac{2}{5}$) $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -\frac{2}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} \end{array}\right]$

$((A+A^T)^{-1})^T = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -\frac{2}{5} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$ $\quad \begin{vmatrix} -2 & 1 & 0 \\ 1 & -\frac{2}{5} & 0 \\ 0 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2}\begin{vmatrix} -2 & 1 \\ 1 & -\frac{2}{5} \end{vmatrix} - 0 + 0 = \frac{1}{2}\left[(-2)(-\frac{2}{5}) - (1)(1)\right] = \frac{1}{2}\left(\frac{4}{5} - 1\right) = \boxed{-\frac{1}{10}}$

part c on back of page

$$X_3 = \frac{|B_3|}{|B|} = \frac{\begin{vmatrix} -2 & 0 & P \\ 0 & -1 & Q \\ 2 & 1 & R \end{vmatrix}}{\begin{vmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}} = \frac{(-2)\begin{vmatrix} -1 & Q \\ 1 & R \end{vmatrix} - 0 + (2)\begin{vmatrix} 0 & P \\ -1 & Q \end{vmatrix}}{(-2)\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - 0 + 0}$$

$$= \frac{-2\left[(-1)(R)-(1)(Q)\right] + 2\left[(0)(Q)-(-1)(P)\right]}{-2\left[(-1)(2)-(1)(1)\right]} = \frac{-2(-R-Q)+2(P)}{-2(-2-1)}$$

$$= \frac{2R+2Q+2P}{6} = \boxed{\frac{R+Q+P}{3}} = X_3$$

Theorem (Wronskian test). Wronskian determinant of f_1, f_2, f_3 nonzero at some invented $x = x_0$ implies independence of f_1, f_2, f_3.
Theorem (Sampling test). Functions f_1, f_2, f_3 are independent if a sampling matrix constructed for some invented samples x_1, x_2, x_3 has nonzero determinant.

Problem 3. (Chapters 1 to 4: 100 points) Independence Tests for Functions.

Let V be the vector space of all functions on $(-\infty, \infty)$. Define functions in V by the equations $f_1(x) = 5 + e^x$, $f_2(x) = 2x$, $f_3(x) = x + x^2$.

A

(a) [50%] Construct the Wronskian matrix W of the given functions f_1, f_2, f_3, then invent a value for x such that $|W| \neq 0$.

A

(b) [50%] Construct a sampling matrix S for the given functions f_1, f_2, f_3, using invented samples x_1, x_2, x_3, such that $|S| \neq 0$.

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a) $f_1(x) = 5+e^x \quad f_2(x) = 2x \quad f_3(x) = x+x^2$

$f_1'(x) = e^x \quad f_2'(x) = 2 \quad f_3'(x) = 1+2x$

$f_1''(x) = e^x \quad f_2''(x) = 0 \quad f_3''(x) = 2$

$$|W| = \begin{vmatrix} 5+e^x & 2x & x+x^2 \\ e^x & 2 & 1+2x \\ e^x & 0 & 2 \end{vmatrix}$$

$x=0$

$$|W| = \begin{vmatrix} 5+e^0 & 2(0) & (0)+(0)^2 \\ e^0 & 2 & 1+2(0) \\ e^0 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 6 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} = 6\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} - 0 + 0 = 6[(2)(2)-(1)(0)]$$
$$= 24 \neq 0$$

✓

b)
$$|S| = \begin{vmatrix} f_1(x_1) & f_2(x_1) & f_3(x_1) \\ f_1(x_2) & f_2(x_2) & f_3(x_2) \\ f_1(x_3) & f_2(x_3) & f_3(x_3) \end{vmatrix}$$

$$|S| = \begin{vmatrix} f_1(1) & f_2(1) & f_3(1) \\ f_1(0) & f_2(0) & f_3(0) \\ f_1(-1) & f_2(-1) & f_3(-1) \end{vmatrix} = \begin{vmatrix} 5+e & 2 & 2 \\ 6 & 0 & 0 \\ 5+e^{-1} & -2 & 0 \end{vmatrix}$$

$$= 2\begin{vmatrix} 6 & 0 \\ 5+e^{-1} & -2 \end{vmatrix} - 0 + 0 = 2[(6)(-2) - (0)(5+e^{-1})] = 2(-12-0) = -24 \neq 0$$ ✓

Rank test. Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3.

Determinant test. Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant.

Pivot test. Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix A has 3 pivot columns.

Orthogonality test. A set of nonzero pairwise orthogonal vectors is independent.

Problem 4. (Chapters 1 to 4: 100 points) Independence Tests for Column Vectors.

Let V be the vector space of all functions on $(-\infty, \infty)$. It is known that the functions $g_1(x) = 5 + e^x$, $g_2(x) = 2x - e^x$, $g_3(x) = e^x$ are independent in V. Let $S = \mathbf{span}(g_1, g_2, g_3)$. Define a coordinate map isomorphism from S to $\mathcal{R}^3$ by

$$T \ : \ c_1(5 + e^x) + c_2(2x - e^x) + c_3(e^x) \quad \text{maps into} \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

A

(a) [60%] Define functions in V by the equations $f_1(x) = 5 + e^x$, $f_2(x) = 5 + 2x$, $f_3(x) = 2x$. Determine the column vectors $\vec{v}_1 = T(f_1), \vec{v}_2 = T(f_2), \vec{v}_3 = T(f_3)$.

A

(b) [40%] Because T is one-to-one and onto, then the given functions f_1, f_2, f_3 are independent in S if and only if the column vectors $T(f_1), T(f_2), T(f_3)$ are independent in $\mathcal{R}^3$. Show details for one of the above independence tests applied to the column vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ calculated in part (a) above.

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a) $\vec{v}_1 = T(f_1) = \left[\begin{array}{ccc|c} 5 & 0 & 0 & 5 \\ 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 1 \end{array}\right]$ mult(1, $\frac{1}{5}$) mult(2, $\frac{1}{2}$) combo(1, 3, $-\frac{1}{5}$) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{bmatrix}$ combo(2, 3, 1) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\quad \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}$

$\vec{v}_2 = T(f_2) = \left[\begin{array}{ccc|c} 5 & 0 & 0 & 5 \\ 0 & 2 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{array}\right]$ mult(1, $\frac{1}{5}$) mult(2, $\frac{1}{2}$) combo(1, 3, $-\frac{1}{5}$) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & -1 \end{bmatrix}$ combo(2, 3, 1) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ $\quad \boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$

$\vec{v}_3 = T(f_3) = \left[\begin{array}{ccc|c} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 \\ 1 & -1 & 1 & 0 \end{array}\right]$ mult(1, $\frac{1}{5}$) mult(2, $\frac{1}{2}$) combo(1, 3, $-\frac{1}{5}$) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$ combo(2, 3, 1) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ $\quad \boxed{\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}$

b) Determinant test:

$$|v_1, v_2, v_3| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = (1)(1)(1) = 1 \neq 0 \ \checkmark$$

DEFINITION. Subset S of vector space V is a subspace of V provided (1), (2), (3) hold:
(1) S contains vector $\vec{0}$.
(2) If $\vec{x}$ and $\vec{y}$ are in S, then $\vec{x}+\vec{y}$ is in S.
(3) If c is a constant and $\vec{x}$ is in S, then $c\vec{x}$ is in S.

Problem 5. (Chapters 2, 4 and 6: 100 points) Subspace Definition and Theorems.

A

(a) [50%] Set S consists of all vectors $\vec{x}$ with components x_1, x_2, x_3 in vector space $\mathcal{R}^3$ such that $x_1 + 2x_3 = x_2$ and $x_1 + x_3 = 0$. Supply proof details which verify that S is a subspace of $\mathcal{R}^3$. Cite a theorem or else verify conditions (1), (2), (3) in the DEFINITION.

A

(b) [50%] Set S consists of all vectors $\vec{x}$ in vector space $\mathcal{R}^4$ which are linear combinations of the vectors

$$\vec{v}_1 = \begin{pmatrix}1\\1\\1\\7\end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix}1\\-1\\0\\1\end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix}2\\0\\1\\8\end{pmatrix}.$$

Supply proof details which verify that S is a subspace of $\mathcal{R}^4$. Cite a theorem or else verify conditions (1), (2), (3) in the DEFINITION.

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a) $x_1 + 2x_3 = x_2$ $\qquad x_1 - x_2 + 2x_3 = 0$
$x_1 + x_3 = 0$ $\qquad x_1 \quad + x_3 = 0$

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} = \vec{0}$$

By the kernel theorem, S is a subspace of $\mathbb{R}^3$ because all values of $\vec{x}$ give the zero funct.

b) (1) $0\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = \vec{0}$ $\therefore$ S contains vector $\vec{0}$ ✓

(2) $\vec{v}_1$ is in S an $\vec{v}_2$ is in S so $\vec{v}_1 + \vec{v}_2$ must be in S, It is because $\vec{v}_1 + \vec{v}_2 + 0\vec{v}_3 = \vec{v}_1 + \vec{v}_2$ ✓

3) $\vec{v}_1$ is in S and $-2\vec{v}_1$ is in S because $-2\vec{v}_1 + 0\vec{v}_2 + 0\vec{v}_3 = -2\vec{v}_1$ ✓

Span $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ or Set S is a subspace of $\mathbb{R}^4$.

Problem 6. (Chapter 6: 100 points) Gram-Schmidt Orthogonalization Process.

Let S be the subspace of $\mathcal{R}^4$ spanned by the independent vectors

$$\vec{\mathbf{x}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{\mathbf{x}}_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{\mathbf{x}}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

A

Find a Gram-Schmidt orthogonal basis $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$ for subspace S.

$$\vec{X}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \quad |\vec{x}_1| = \sqrt{1^2+1^2+1^2+0^2} = \sqrt{3} \qquad \boxed{\vec{V}_1 = \frac{\vec{X}_1}{|\vec{X}_1|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ 0 \end{bmatrix}}$$

$$\vec{U}_2 = \vec{X}_2 - \frac{\vec{X}_2 \cdot \vec{X}_1}{\vec{X}_1 \cdot \vec{X}_1} \vec{X} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \cancel{\frac{-1+1+0+0}{1+1+1+0}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad |\vec{U}_2| = \sqrt{(-1)^2+1^2+0^2+1^2} = \sqrt{3}$$

0

$$\frac{\vec{U}_2}{|\vec{U}_2|} = \boxed{\vec{V}_2 = \begin{bmatrix} \frac{-1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{0}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}}$$

$$\vec{U}_3 = X_3 - \frac{\vec{X}_3 \cdot \vec{X}_1}{X_1 \cdot X_1} X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{1+1+0+0}{1+1+1+0} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

$$\vec{W}_3 = \vec{U}_3 - \frac{\vec{U}_3 \cdot \vec{X}_2}{X_2 \cdot X_2} X_2 = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} - \frac{-\frac{1}{3}+\frac{1}{3}+0+1}{1+1+0+1} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ 0 \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$$

$$|\vec{W}_3| = \sqrt{\frac{4}{9}+0+\frac{4}{9}+\frac{4}{9}} = \sqrt{\frac{12}{9}} = \frac{2\sqrt{3}}{3}$$

$$\frac{\vec{W}_3}{|\vec{W}_3|} = \boxed{\vec{V}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ 0 \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}}$$

Problem 7. (Chapters 1 to 6: 100 points) Symmetric Matrices and the Invertible Matrix Theorem.

Let A be an $m \times n$ matrix and assume that $A^T A$ is invertible. Prove that the columns of A are linearly independent.

Expected: A referenced result from *"The Invertible Matrix Theorem"* should appear as a precisely stated LEMMA, the proof of the LEMMA deferred to the textbook.

A

$A\vec{x}=0$ left multiply both sides by A^T $A^T A\vec{x}=0$

Details lacking context & Continuity.

$A^T A$ is invertible $\therefore$ by the invertible matrix theorem the colu of $A^T A$ are linearly independent $\therefore$ $\vec{x}=\vec{0}$ and since $\vec{x}=\vec{0}$ that means $A\vec{x}=\vec{0}$ only with the tri solution $\vec{x}=\vec{0}$ as well which means that the colu of A are linearly independent.

ok

Problem 7. (Chapters 1 to 6: 100 points) Symmetric Matrices and the Invertible Matrix Theorem.

Let A be an $m \times n$ matrix and assume that $A^T A$ is invertible. Prove that the columns of A are linearly independent.

Expected: A referenced result from *"The Invertible Matrix Theorem"* should appear as a precisely stated LEMMA, the proof of the LEMMA deferred to the textbook.

A-

If A^TA is invertible, then the nullspace of A^TA is $\vec{0}$. The goal is to prove the nullspace of A is $\vec{0}$ and therefore the columns of A are linearly independent.

Take $A^TA\vec{x} = 0 \Rightarrow \vec{x} = 0$ $\quad x^TA^TAx = x^T0$

If A^TA is invertible, $\|A^T\| = \|A\|$

$\|A\|^2 = \vec{0}$

$\|A\| = \vec{0}$

A should be $A\vec{x}$

Therefore the nullspace of A is $\vec{0}$ and A is linearly independent

Problem 8. (Chapter 5: 100 points) Eigenanalysis and Diagonalization.

The matrix A below has eigenvalues 2, 8 and 8.

$$A = \begin{pmatrix} 12 & 4 & -1 \\ -4 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

A **(a)** [80%] Compute all eigenpairs of A.

Expected: For each eigenvalue λ, first compute the RREF of $A - \lambda I$, then compute all eigenvectors for λ (they are Strang's solutions).

A **(b)** [20%] Is A diagonalizable? Explain why or why not.

$\frac{4}{15}$ $\frac{8}{30}$

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a) $\lambda = 2$ $\begin{bmatrix} 12-2 & 4 & -1 & 0 \\ -4 & 4-2 & -2 & 0 \\ 0 & 0 & 2-2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 10 & 4 & -1 & 0 \\ -4 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ mult$(1, \frac{1}{10})$ combo$(1, 2, \frac{4}{10})$ $\begin{bmatrix} 1 & \frac{2}{5} & -\frac{1}{10} & 0 \\ 0 & \frac{18}{5} & -\frac{12}{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ combo$(2, 1, -\frac{1}{9})$ mult$(2, \frac{5}{18})$

$\begin{bmatrix} 1 & 0 & \frac{1}{6} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $x_1 + \frac{1}{6}x_3 = 0$, $x_2 - \frac{2}{3}x_3 = 0$, $x_3 = x_3$ $\quad x_1 = -\frac{1}{6}x_3$, $x_2 = \frac{2}{3}x_3$ $\quad \boxed{\vec{v}_1 = \begin{bmatrix} -1 \\ 4 \\ 6 \end{bmatrix}}$ ✓

$\lambda = 8$ $\begin{bmatrix} 12-8 & 4 & -1 & 0 \\ -4 & 4-8 & -2 & 0 \\ 0 & 0 & 2-8 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 4 & -1 & 0 \\ -4 & -4 & -2 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$ mult$(1, \frac{1}{4})$ combo$(1, 2, 1)$ $\begin{bmatrix} 1 & 1 & -\frac{1}{4} & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & -6 & 0 \end{bmatrix}$ combo$(2, 1, -\frac{1}{12})$ mult$(2, -\frac{1}{3})$ combo$(2, 3, -2)$ $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_2 = 0$, $x_2 = x_2$, $x_3 = 0$ $\quad x_1 = -x_2$ $\quad \boxed{\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}$ ✓

b) A is not diagonizable because the eigenvalue 8 is repeated but only has one eigenvector. In other words, A is a 3×3 matrix but only has two eigen pairs, and it needs three to be diagonizable.

Problem 9. (Chapter 6: 100 points) Near Point Theorem and Least Squares.

A **(a)** [60%] Define $A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. Write the normal equations for the inconsistent problem $A\vec{x} = \vec{b}$ and solve for the least squares solution $\hat{x}$.

A **(b)** [20%] Least squares can be used to find the best fit line $y = mx + b$ for the (x, y)-data points $(-1, 3)$, $(0, 1)$, $(1, 2)$. Find the line equation by the method of least squares.

Expected: The matrix A you create for part (b) should match the matrix A of part (a). Save time by using the computations from (a).

A **(c)** [20%] Continue part (a). Compute vector $\hat{b} = A\hat{x}$, which is the near point to $\vec{b}$ in the column space of A. Then compute the mean square error, which is the norm of the vector $\vec{b} - \hat{b}$.

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a) $A^T A \hat{x} = A^T \vec{b}$ $\quad \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \hat{x} = \begin{bmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ $\quad \begin{bmatrix} 1+1 & -1+1 \\ -1+1 & 1+1+1 \end{bmatrix} \hat{x} = \begin{bmatrix} -3+2 \\ 3+1+2 \end{bmatrix}$

$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \hat{x} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$ $\quad \left[\begin{array}{cc|c} 2 & 0 & -1 \\ 0 & 3 & 6 \end{array}\right] \begin{array}{l} \text{mult}(1, \frac{1}{2}) \\ \text{mult}(2, \frac{1}{3}) \end{array}$ $\left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 2 \end{array}\right]$ $\quad \boxed{\hat{x} = \begin{bmatrix} -\frac{1}{2} \\ 2 \end{bmatrix}}$

b) $A\vec{x} = \vec{b}$ $\quad A = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad \vec{x} = \begin{bmatrix} m \\ b \end{bmatrix}$ $\quad \vec{b} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

using part (a) calculations $\hat{x} = \begin{bmatrix} -\frac{1}{2} \\ 2 \end{bmatrix}$ so $\boxed{y = -\frac{1}{2}x + 2}$

c) $\hat{b} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + 2 \\ 0 + 2 \\ -\frac{1}{2} + 2 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{5}{2} \\ 2 \\ \frac{3}{2} \end{bmatrix}}$

$\|\vec{b} - \hat{b}\| = \left\| \begin{bmatrix} 3 - \frac{5}{2} \\ 1 - 2 \\ 2 - \frac{3}{2} \end{bmatrix} \right\| = \left\| \begin{bmatrix} \frac{1}{2} \\ -1 \\ \frac{1}{2} \end{bmatrix} \right\| = \sqrt{(\frac{1}{2})^2 + (-1)^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{4} + 1 + \frac{1}{4}} = \sqrt{\frac{6}{4}} = \boxed{\frac{\sqrt{6}}{2}}$

Problem 10. (Chapter 7: 100 points) Spectral Theorem and $AQ = QD$.

The spectral theorem says that a symmetric matrix A satisfies $AQ = QD$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 7 & 3 \\ 3 & 7 \end{pmatrix}$.

$\begin{vmatrix} 7-\lambda & 3 \\ 3 & 7-\lambda \end{vmatrix} = 0 \quad (7-\lambda)(7-\lambda) - (3)(3) = 0 \quad \lambda^2 - 14\lambda + 49 - 9 = 0 \quad \lambda^2 - 14\lambda + 40 = 0$

$(\lambda - 4)(\lambda - 10) = 0 \quad \lambda = 10, 4$

$\lambda = 10$ // $\left[\begin{array}{cc|c} 7-10 & 3 & 0 \\ 3 & 7-10 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc} -3 & 3 & 0 \\ 3 & -3 & 0 \end{array}\right]$ mult$(1, \frac{-1}{3})$ combo$(1, 2, 1)$ $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 - x_2 = 0$ $x_2 = x_2$ $x_1 = x_2$

$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\frac{\vec{v}_1}{|\vec{v}_1|} = \vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$|\vec{v}_1| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\lambda = 4$ // $\left[\begin{array}{cc|c} 7-4 & 3 & 0 \\ 3 & 7-4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 3 & 3 & 0 \\ 3 & 3 & 0 \end{array}\right]$ mult$(1, \frac{1}{3})$ combo$(1, 2, -1)$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1 + x_2 = 0$ $x_2 = x_2$ $x_1 = -x_2$

$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\frac{\vec{v}_2}{|\vec{v}_2|} = \vec{v}_2 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$|\vec{v}_2| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

$Q = \begin{bmatrix} \vec{v}_1, \vec{v}_2 \end{bmatrix}$

$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$\boxed{Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \qquad D = \begin{bmatrix} 10 & 0 \\ 0 & 4 \end{bmatrix}}$$

Problem 11. (Chapter 7: 100 points) Singular Value Decomposition.

Determine the singular value decomposition for the matrix $A = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$.

A

$A = U\Sigma V^T$ Find singular values using A^TA:

$$\begin{array}{r} 1 \\ 24 \\ \times 24 \\ \hline 96 \\ 480 \\ \hline 576 \end{array}$$

$$A^TA = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix} = \begin{pmatrix} 40 & 24 \\ 24 & 40 \end{pmatrix}$$

$$(A^TA - \lambda I) = \begin{pmatrix} 40-\lambda & 24 \\ 24 & 40-\lambda \end{pmatrix}$$

characteristic equation

$$(40-\lambda)(40-\lambda) - 576$$

$$\lambda^2 - 80\lambda + 1600 - 576$$

$$\lambda - 80\lambda + 1024$$

$$(\lambda-16)(\lambda-64)$$

$$\lambda_2 = 16 \quad \lambda_1 = 64$$

$$\begin{array}{r} 40 \\ 40 \\ \hline 1600 \end{array}$$

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{64} = 8$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{16} = 4$$

For $\lambda = 64$ $(A^TA - 64I)$

$$\begin{pmatrix} -24 & 24 \\ 24 & -24 \end{pmatrix} \sim \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x_1 = x_2 \quad v_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$\begin{array}{r} 2 \\ 16 \\ 64 \\ \hline 64 \\ 960 \\ \hline 1024 \end{array}$$

For $\lambda = 16$ $A^TA - 16I$

$$\begin{pmatrix} 24 & 24 \\ 24 & 24 \end{pmatrix} \sim \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \quad x_1 = -x_2 \quad v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

Find $u_1 = \dfrac{Av_1}{\sigma_1} = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \div 8 = \begin{pmatrix} 8/\sqrt{2} \\ 8/\sqrt{2} \end{pmatrix} \div 8 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$

$$u_2 = \frac{Av_2}{\sigma_2} = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}\begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \div 4 = \begin{pmatrix} -4/\sqrt{2} \\ 4/\sqrt{2} \end{pmatrix} \div 4 = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$A = U\Sigma V^T = \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{U}\underbrace{\begin{pmatrix} 8 & 0 \\ 0 & 4 \end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{V^T}$$

Answer check

$$\begin{pmatrix} 8/\sqrt{2} & 8/\sqrt{2} \\ -4/\sqrt{2} & 4/\sqrt{2} \end{pmatrix} \quad \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$$

Problem 11. (Chapter 7: 100 points) Singular Value Decomposition.

Determine the singular value decomposition for the matrix $A = \begin{pmatrix} 6 & 2 \\ 2 & 6 \end{pmatrix}$.

A-

$1600 - 576 = 1024$

$A^TA = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 36+4 & 12+12 \\ 12+12 & 4+36 \end{bmatrix} = \begin{bmatrix} 40 & 24 \\ 24 & 40 \end{bmatrix}$

$\begin{vmatrix} 40-\lambda & 24 \\ 24 & 40-\lambda \end{vmatrix} = 0 \quad (40-\lambda)(40-\lambda) - (24)(24) = \lambda^2 - 80\lambda + 1600 - 576 = 0 \quad \lambda^2 - 80 + 1024 = 0 \quad (\lambda-16)(\lambda-64) = 0$

$\lambda_2 = 16 \quad \lambda_1 = 64$

$\sigma_2 = \sqrt{\lambda_2} = 4 \quad \sigma_1 = \sqrt{\lambda_1} = 8$

$\lambda = 64 \,//\, \begin{bmatrix} 40-64 & 24 & 0 \\ 24 & 40-64 & 0 \end{bmatrix} \to \begin{bmatrix} -24 & 24 & 0 \\ 24 & -24 & 0 \end{bmatrix} \begin{matrix} \text{mult}(1, -\frac{1}{24}) \\ \text{combo}(1,2,1) \end{matrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} x_1 - x_2 = 0 \\ x_2 = x_2 \end{matrix} \quad x_1 = x_2$

$\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \frac{\vec{w}_1}{|\vec{w}_1|} = \vec{v}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$|\vec{w}_1| = \sqrt{2}$

$\lambda = 16 \,//\, \begin{bmatrix} 40-16 & 24 & 0 \\ 24 & 40-16 & 0 \end{bmatrix} \to \begin{bmatrix} 24 & 24 & 0 \\ 24 & 24 & 0 \end{bmatrix} \begin{matrix} \text{mult}(1, \frac{1}{24}) \\ \text{combo}(1,2,-1) \end{matrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} x_1 + x_2 = 0 \\ x_2 = x_2 \end{matrix} \quad x_1 = -x_2$

$\vec{w}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \frac{\vec{w}_2}{|\vec{w}_2|} = \vec{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$

$|\vec{w}_2| = \sqrt{2}$

$\vec{u}_1 = A\vec{v}_1 = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{8}{\sqrt{2}} \\ \frac{8}{\sqrt{2}} \end{bmatrix}$

$\vec{u}_2 = A\vec{v}_2 = \begin{bmatrix} 6 & 2 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} -\frac{4}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} \end{bmatrix}$

$\frac{A\vec{v}_2}{\sigma_2} = \frac{1}{4}\begin{bmatrix} \vdots \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$

$U \Sigma V^T = \boxed{\begin{bmatrix} \frac{8}{\sqrt{2}} & -\frac{4}{\sqrt{2}} \\ \frac{8}{\sqrt{2}} & \frac{4}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 8 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}}$

not $\perp$ matrix

Need $U^TU = I$

ok $V^TV = I$

Problem 12. (Chapter 4: 100 points) Linear Transformations as Matrix Multiply.

Let the linear transformation T from $\mathcal{R}^2$ to $\mathcal{R}^2$ be defined by its action on two independent vectors:

$$T\left(\begin{pmatrix}2\\3\end{pmatrix}\right) = \begin{pmatrix}4\\3\end{pmatrix}, \quad T\left(\begin{pmatrix}1\\2\end{pmatrix}\right) = \begin{pmatrix}4\\1\end{pmatrix}.$$

A

Find the unique 2×2 matrix A such that T is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$.

$T\left(\begin{bmatrix}2\\3\end{bmatrix}\right) = \begin{bmatrix}4\\3\end{bmatrix} \quad T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}4\\1\end{bmatrix}$

$A\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix} = \begin{bmatrix}4 & 4\\3 & 1\end{bmatrix}$

$\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}^{-1} = \frac{1}{4-3}\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix} = \begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}$

$A\begin{bmatrix}2 & 1\\3 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix} = \begin{bmatrix}4 & 4\\3 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\-3 & 2\end{bmatrix}$

$A\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}8-12 & -4+8\\6-3 & -3+2\end{bmatrix}$

$$\boxed{A = \begin{bmatrix}-4 & 4\\3 & -1\end{bmatrix}}$$

Problem 13. (Chapters 4 and 6: 100 points) Orthogonality.

Let symbols a, b, c, d, e, f represent certain real numbers. Define $A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$. Define subspaces

$$S_1 = \text{the column space of the transpose matrix } A^T = \mathbf{Col}(A^T)$$
$$S_2 = \text{the null space of } A = \mathbf{Null}(A).$$

Let $\vec{\mathbf{x}}$ belong to S_1 and let $\vec{\mathbf{y}}$ belong to S_2. Prove that their dot product is zero: $\vec{\mathbf{x}} \cdot \vec{\mathbf{y}} = 0$.

Expected: Apply the definition of matrix multiply in terms of dot products. No theorems are used, only definitions.

A

$\vec{y}$ belongs to S_2 so $A\vec{y} = \vec{0}$ which means that

$ay_1 + by_1 + cy_1 = 0$

$dy_2 + ey_2 + fy_2 = 0$

$\vec{x}$ belongs to S_1 so $A^T = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$, assuming $\vec{x}$ exists it must be some linear combination of $\begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} d \\ e \\ f \end{bmatrix}$ so $\vec{x} = c_1 \begin{bmatrix} a \\ b \\ c \end{bmatrix} + c_2 \begin{bmatrix} d \\ e \\ f \end{bmatrix}$

so $\vec{x} \cdot \vec{y} = c_1 ay_1 + c_1 by_1 + c_1 cy_1 + c_2 dy_2 + c_2 ey_2 + c_2 fy_2$

$= c_1 (ay_1 + by_1 + cy_1) + c_2 (dy_2 + ey_2 + fy_2)$

using identity created above.

$= c_1 (0) + c_2 (0) = 0$ ✓

Problem 14. (Chapters 1 to 7: 100 points) Fundamental Theorem of Linear Algebra.

(a) [40%] Give a technical definition for each of the four fundamental subspaces appearing in the Fundamental Theorem of Linear Algebra. With each definition, describe how to compute a basis for the subspace.

(b) [30%] What are the dimensions of the four subspaces?

A

(c) [30%] State the orthogonality relations for the four fundamental subspaces.

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a) Col(A) is the column space of the matrix A which a basis is formed by the pivot columns of the matrix A

Row(A) is the row space of matrix A which is the column space of matrix A^T. The basis is formed by the pivot columns of matrix A^T

Null(A) is the set of $\vec{x}$ such that $A\vec{x}=\vec{0}$. The basis is formed by Strang's solutions to $A\vec{x}=\vec{0}$

Null(A^T) is the set of $\vec{x}$ such that $A^T\vec{x}=\vec{0}$. The basis is formed by Strang's solutions to $A^T\vec{x}=\vec{0}$

b) Col(A) has dimensions r
Row(A) has dimensions r
Null(A) has dimensions m-r
Null(A^T) has dimensions n-r

for a matrix nxm with rank r

c) Col(A) is orthogonal to Null(A)
Row(A) is orthogonal to Null(A^T)