Math 2270-2 Spring 2017 Solution key

Midterm 2

1000 points / 10 problems

Problem 1. (100 points) Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} z_1 & z_2 & 0 \\ 0 & z_3 & 0 \\ 1 & z_4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, display the formula for x_2 according to Cramer's Rule. To save time, **do not compute determinants!**

A

$$x_2 = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} z_1 & -3 & 0 \\ 0 & 5 & 0 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} z_1 & z_2 & 0 \\ 0 & z_3 & 0 \\ 1 & z_4 & 1 \end{vmatrix}}$$

Problem 2. (100 points) Define matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & 1 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

A

$$\underline{U} \qquad\qquad \underline{L}$$

$$\begin{pmatrix} 2 & 1 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix}$$

combo(-3,1,2)

combo(-4,1,3)

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 10 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & & 1 \end{pmatrix}$$

combo (-2,2,3)

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix}$$

$$A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Problem 3. **(100 points)** Find the complete vector solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

A

Please display vector answers for both $\vec{x}_h$ and $\vec{x}_p$. The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

$$\left(\begin{array}{ccccc|c} 0 & 3 & 1 & 0 & 0 & 3 \\ 0 & 3 & 3 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 0 & 3 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 0 & 3 & 0 & -1/2 & -1/2 & 4 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 0 & 1 & 0 & -1/6 & -1/6 & 4/3 \\ 0 & 0 & 1 & 1/2 & 1/2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_2 = 1/6\, x_4 + 1/6\, x_5 + 4/3 \\ x_3 = -1/2\, x_4 - 1/2\, x_5 - 1 \\ x_1, x_4, x_5 \text{ are free} \end{cases}$$

$$\vec{x} = \underbrace{x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1/6 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} 0 \\ 1/6 \\ -1/2 \\ 0 \\ 1 \end{pmatrix}}_{\vec{x}_h} + \underbrace{\begin{pmatrix} 0 \\ 4/3 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{x}_p}$$

Problem 4. (100 points) Let V be the vector space of all functions on $(-\infty, \infty)$. Define subspace $S = \textbf{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent vectors defined respectively by the equations $y = x - 1$, $y = 1 + x^2$, $y = 2x + x^2$. If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, then the uniquely determined constants c_1, c_2, c_3 are called the *coordinates of $\vec{v}$ relative to the basis* $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

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Compute c_1, c_2, c_3 for $\vec{v}$ defined by $y = 1 + 2x + 3x^2$

let $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$A\vec{x} = \vec{y}$

$$\left(\begin{array}{ccc|c} -1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right) \sim \begin{pmatrix} -1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 3 \\ 1 & 0 & 2 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & -1 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$C_1 = 2$

$C_2 = 3$

4 $C_3 = 0$

Problem 5. **(100 points)** The functions 1, x^2, x^5 represent independent vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ in the vector space V of all functions on $0 < x < \infty$. The set $S =$ **span**$(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a subspace of V. Let vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in V be defined by the functions $1+x^2, x^5-x^2, 5+2x^5$, respectively. The **coordinate map** defined by

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$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

maps the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ into the following images in $\mathcal{R}^3$, respectively:

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}.$$

The independence tests below can decide independence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by formulating the independence question in vector space V or in vector space $\mathcal{R}^3$, because the coordinate map takes independent sets to independent sets.

Check below all independence tests which apply to decide independence of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

$f_1 = 1$
$f_2 = x^2$
$f_3 = x^3$

$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ $\vec{v}_3 = \begin{pmatrix} 5 \\ 0 \\ 2 \end{pmatrix}$

☑	**Wronskian test**	Wronskian determinant of f_1, f_2, f_3 nonzero at $x = x_0$ implies independence of f_1, f_2, f_3.
☑	**Sampling test**	Sampling determinant for samples $x = x_1, x_2, x_3$ nonzero implies independence of f_1, f_2, f_3.
☑	**Rank test**	Three vectors are independent if their augmented matrix has rank 3.
☑	**Determinant test**	Three vectors are independent if their augmented matrix is square and has nonzero determinant.
☐	**Orthogonality test**	Three vectors are independent if they are all nonzero and pairwise orthogonal.
☑	**Pivot test**	Three vectors are independent if their augmented matrix A has 3 pivot columns.

Problem 6. (100 points) The matrix $A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ -1 & -1 & 5 \end{pmatrix}$ has eigenvalues $3, 4, 4$.

(a) [80%] Find all eigenvectors for $\lambda = 4$. To save time, **don't find $\lambda = 3$ eigenvectors**.

(b) [20%] Report whether or not matrix A is diagonalizable. Explain.

a)

$$A-4I = \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow{R_2 += -R_1} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 += -R_1} \begin{pmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_3 = t_2$
$x_2 = t_1$
$x_1 = t_2 - t_1$

$$\vec{x} = t_1 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \vec{v}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

b) Yes, since A has 3 linearly independent eigenvectors it follows that it is diagonalizable.

Problem 7. (100 points) Define S to be the set of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = x_2$ and $x_3x_2 = x_1x_2$. Show that S is **NOT** a subspace of $\mathcal{R}^3$, that is, exhibit a counterexample to one of the items in the *Subspace Criterion*.

A

Let $x_1 = 1, x_2 = 0$ $x_3 = -1$, $1 - 1 = 0$ ✓ $(-1)(0) = (1)(0)$

Let $x_1 = 1, x_2 = 2, x_3 = 1$ $1 + 1 = 2$ ✓ $(1)(2) = (1)(2)$

Since $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ is a solution and $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ is a solution then suppose their sum is a solution. $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$ $2 + 0 = 2$ ✓

$2(2) \neq 2(0)$

Therefore, S is not closed under addition.

Problem 8. (100 points) Let A be a 4×3 matrix. Assume the columns of $A^T A$ are dependent. Prove or disprove that A has dependent columns.

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Let $A^T A \vec{x} = \vec{0}$

Then $\vec{x}^T A^T A \vec{x} = \vec{x}^T \vec{0} \Longrightarrow (Ax)^T A\vec{x} = \vec{0}$

$\Longrightarrow A\vec{x} \cdot A\vec{x} = \vec{0} \Longrightarrow \|A\vec{x}\|^2 = \vec{0} \Longrightarrow A\vec{x} = \vec{0}$

Since $A^T A \vec{x} = \vec{0}$ and $A\vec{x} = \vec{0}$, the columns of A must be dependent because $A^T A \vec{x} = \vec{0}$ has a non-trivial solution. e.g. $\vec{x}$ contains more than just the zero vector □

Problem 9. (100 points) Let 3×3 matrices A, B and C be related by $AP = PB$ and $BQ = QC$ for some invertible matrices P and Q. Assume B has eigenvalues $2, 3, 7$. Prove that matrices A and C also have eigenvalues $2, 3, 7$.

Because matrices P and Q are invertible, the given equations can be rewritten:

$$A = PBP^{-1}$$
$$B = QCQ^{-1}$$

In order to find the eigenvalues, we subtract λI and solve for when the determinant is 0:

$$|A - \lambda_1 I| = |PBP^{-1} - \lambda_1 I| = 0$$
$$|B - \lambda_2| = |QCQ^{-1} - \lambda_2 I| = 0$$

Because both cases follow identical steps, I display only one here:

$$\begin{aligned} |A - \lambda_1 I| &= |PBP^{-1} - \lambda_1 I| \\ &= |P||B - \lambda_1 I||P^{-1}| \\ &= |PP^{-1}||B - \lambda_1 I| \\ &= |I||B - \lambda_1 I| \\ &= |B - \lambda_1 I| = 0 \end{aligned}$$

Therefore, A and B have the same eigenvalues. By identical reasoning, B and C have the same eigenvalues. By transitivity, A and C have the same eigenvalues, so the eigenvalues of C are also 2, 3, 7. QED

Problem 10. (100 points) The **Fundamental Theorem of Linear Algebra** says that the null space of a matrix is orthogonal to the row space of the matrix.

Let A be an $m \times n$ matrix. Define subspaces S_1 = column space of A, S_2 = null space of A^T. Prove that the only vector $\vec{v}$ in both S_1 and S_2 is the zero vector.

Let $\vec{v}$ be an arbitrary vector in S_1

and let $\vec{u}$ be an arbitrary vector in S_2

Then, by funamental thm of linear algebra

$\vec{v} \perp \vec{u}$ since column space $\perp$ null space of transpose

Then, we have $\vec{v} \cdot \vec{u} = 0$ so it follows that

the only vector, $\vec{v}, \vec{u}$ in <u>both</u> S_1, S_2 is the

zero vector $\square$