Student Solutions Exam 2 2270-2 S2018

Problem 1. **(100 points)** Define matrix A, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -4 & 1 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} -3 \\ 5 \\ 1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

For the system $A\vec{x} = \vec{b}$, display the formula for x_3 according to Cramer's Rule. **To save time, do not compute determinants!**

$$x_i = \frac{\det A_i(\vec{b})}{\det A}$$

$$x_3 = \frac{\begin{vmatrix} -2 & 3 & -3 \\ 0 & -4 & 5 \\ 1 & 4 & 1 \end{vmatrix}}{\begin{vmatrix} -2 & 3 & 0 \\ 0 & -4 & 1 \\ 1 & 4 & 1 \end{vmatrix}}$$

Problem 2. **(100 points)** Define matrix $A = \begin{pmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{pmatrix}$. Find a lower triangular matrix L and an upper triangular matrix U such that $A = LU$.

$$\begin{bmatrix} 2 & 3 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$U = \begin{bmatrix} 2 & 3 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

Problem 3. **(100 points)** Find the complete vector solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$A = \begin{pmatrix} 0 & 3 & 1 & 0 & 0 \\ 0 & 3 & 3 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}.$$

Expected: (a) Augmented matrix. (b) Toolkit steps to RREF. (c) Translation of RREF to scalar equations. (d) Scalar general solution. (e) Find the homogeneous solution $\vec{x}_h$, which is a linear combination of Strang's special solutions. (f) Find a particular solution $\vec{x}_p$. (g) Write the vector general solution $\vec{x} = \vec{x}_h + \vec{x}_p$.

$$\left[\begin{array}{ccccc|c} 0 & 3 & 1 & 0 & 0 & 3 \\ 0 & 3 & 3 & 0 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left(\begin{array}{ccccc|c} 0 & 3 & 1 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 & 6 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 0 & 3 & 0 & 0 & -3 & 4 \\ 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 0 & 1 & 0 & 0 & -1 & 4/3 \\ 0 & 0 & 1 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 & \text{free} \\ x_2 & = 4/3 + x_5 \\ x_3 & = -1 - x_5 \\ x_4 & \text{free} \\ x_5 & \text{free} \end{cases}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_5 + \frac{4}{3} \\ -x_5 - 1 \\ x_4 \\ x_5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 4/3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x} = \underbrace{t_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{x}_h} + \underbrace{\begin{bmatrix} 0 \\ 4/3 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{\vec{x}_p}$$

Problem 4. (100 points)

Definition. If $\vec{b}_1, \vec{b}_2, \vec{b}_3$ are a basis for subspace W of vector space V, and $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3$ is a given linear combination of these vectors, then the uniquely determined constants c_1, c_2, c_3 are called the *coordinates of $\vec{x}$ relative to the basis $\vec{b}_1, \vec{b}_2, \vec{b}_3$*.

Below, let V be the vector space of all functions on $(-\infty, \infty)$. Define subspace $S = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent vectors defined respectively by the equations $y = 1 + x$, $y = 2 + x^2$, $y = 2 + x + x^2$.

A (a) [40%] Let $W = \text{span}\{1, x, x^2\}$. Assume known that $1, x, x^2$ are independent functions. Already, $S = \text{span}\{1 + x, 2 + x^2, 2 + x + x^2\}$ is a subset of W. Prove that W is a subset of S (this proves that $W = S$, therefore $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent).

A (b) [60%] Define vector $\vec{v}$ in S by equation $y = 3 + 4x + x^2$. Compute c_1, c_2, c_3 satisfying the equation $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, using coordinate map methods.

Expected in (b): Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are defined by $1+x, 2+x^2, 2+x+x^2$, respectively. Calculations of c_1, c_2, c_3 are to be done using column vectors from $\mathcal{R}^3$, not functions from V. **Zero credit for not using column vectors.**

(a) Let $\vec{v} \in W$, then $\vec{v}$ is of the form

$c_1 + c_2 x + c_3 x^2$ for some $c_1, c_2, c_3 \in \mathbb{R}$.

Have that any $\vec{s} \in S$ is of the form $c_1(1+x) + c_2(2+x^2) + c_3(2+x+x^2)$

$= c_1 + c_1 x + 2c_2 + c_2 x^2 + 2c_3 + c_3 x + c_3 x^2 = (c_1 + 2c_2 + 2c_3) + (c_1 + c_3)x + (c_2 + c_3)x^2$

for some $c_1, c_2, c_3 \in \mathbb{R}$. Therefore, $\vec{v} \in S$ and $W \subseteq S$.

(b) $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$[\vec{v}]_W = c_1[\vec{v}_1]_W + c_2[\vec{v}_2]_W + c_3[\vec{v}_3]_W$

$[\vec{v}]_W = \left[[\vec{v}_1]_W \; [\vec{v}_2]_W \; [\vec{v}_3]_W \right] (\vec{v})_c$

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -2 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & -2 & -1 & 1 \\ 0 & 0 & \frac{1}{2} & \frac{3}{2} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 2 & 3 \\ 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Therefore, the coordinates are

$$\begin{cases} c_1 & 1 \\ c_2 & -2 \\ c_3 = 3 \end{cases}$$

Problem 5. **(100 points)** The functions $1, x^2, x^4$ represent independent vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ in the vector space V of all functions on $0 < x < \infty$. The set $S = \mathrm{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a subspace of V. Let vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in V be defined by the functions $1 - x^2, x^4 + x^2, 3 + 2x^4$, respectively. The coordinate map defined by

$$c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 \to \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

maps the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ into the following images in $\mathcal{R}^3$, respectively:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}.$$

The independence tests below can decide independence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by formulating the independence question in vector space V or in vector space $\mathcal{R}^3$, because the coordinate map takes independent sets to independent sets.

Check below all independence tests which apply to decide independence of $\vec{v}_1, \vec{v}_2, \vec{v}_3$. Explain how each checked test applies, giving details/reasons. Zero credit for checking a box without explanation.

$$|W| = \begin{vmatrix} 1-x^2 & x^4+x^2 & 3+2x^4 \\ -2x & 4x^3+2x & 8x^3 \\ -2 & 12x^2+2 & 24x^2 \end{vmatrix} \qquad |W(0)| = \begin{vmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ -2 & 2 & 0 \end{vmatrix} = 3\begin{vmatrix} 0 & 0 \\ -2 & 2 \end{vmatrix}$$

$$|W(1)| = \begin{vmatrix} 0 & 2 & 5 \\ -2 & 6 & 8 \\ -2 & 14 & 24 \end{vmatrix} = -2\begin{vmatrix} -2 & 8 \\ -2 & 24 \end{vmatrix} + 5\begin{vmatrix} -2 & \\ -2 & \end{vmatrix}$$

$$= \underset{-32}{-2(-48+16)} \quad \underset{-12}{5(-24+12)} = 64 - 60 = 4$$

functions

A ☑ **Wronskian test** — Nonzero Wronskian determinant of f_1, f_2, f_3 at invented value $x = x_0$ implies independence of f_1, f_2, f_3.

Explain: $|W(1)|$ is non-zero

C ☑ **Sampling test** — Nonzero sampling determinant for invented samples $x = x_1, x_2, x_3$ implies independence of f_1, f_2, f_3.

Explain: functions are given for this example ⇐ Sample matrix here, define here, no samples. No det ≠ 0

$$\begin{pmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ 0 & 0 & \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

fixed vectors

A ☑ **Rank test** — Three column vectors are independent if their augmented matrix has rank 3.

Explain: The given fixed vectors have a rank of 3 to display independence Maybe done above?

A ☑ **Determinant test** — Three column vectors are independent if their augmented matrix is square and has nonzero determinant.

Explain: Square matrix and fixed vector available

$$\begin{vmatrix} 1 & 0 & 3 \\ -1 & 1 & 0 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 1 & 2 \end{vmatrix} = (2-0) + (0-3) = 2 - 3 = -1$$

A ☐ **Orthogonality test** — Three column vectors are independent if they are all nonzero and pairwise orthogonal.

Explain: Not orthogonal $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 - 1 + 0 = -1$

B ☑ **Pivot test** — Three column vectors are independent if their augmented matrix A has 3 pivot columns.

Explain: RREF of vectors can produce pivot columns to determine independence. why 3 pivots?

Problem 6. (100 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(5, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right), \quad \left(2+i, \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}\right), \quad \left(2-i, \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}\right).$$

A

(a) [30%] Display an invertible matrix P and a diagonal matrix D such that $AP = PD$. Matrices P and D can contain complex numbers.

B

(b) [30%] Display a real invertible matrix P_1 and a real diagonal matrix D_1 such that $AP_1 = P_1 D_1$. Neither P_1 nor D_1 can contain complex numbers. The construction of D_1 uses the map $a + ib \to \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

A

(c) [40%] Display a matrix product formula for A in which the factors contain only real numbers. To save time, do not evaluate any matrix products.

(a) A has 3 linearly independent eigenvectors, so by the diagonalization theorem

$AP = PD$ where $P = \begin{bmatrix} 1 & i & -i \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}$

(B) $D_1 = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$ Has to be 3×3

(c) Since $AP_1 = P_1 D \Rightarrow A = P_1 D P_1^{-1}$

Problem 7. (100 points)

Definition: A subset S of a vector space V is a subspace of V provided

(1) The zero vector is in S

(2) If vectors $\vec{x}$ and $\vec{y}$ are in S, then $\vec{x} + \vec{y}$ is in S.

(3) If vector $\vec{x}$ is in S and c is any scalar, then $c\vec{x}$ is in S.

A

Let V be the vector space of all real-valued functions on $(-\infty, \infty)$. Invent an example of a nonvoid subset S of V that satisfies two of the items in the above definition of subspace, but fails the third item.

Let S be the set of functions defined on $[0, \infty]$ and where f is ~~positive~~. NON-Negative

then $\vec{0} \in S$, $\vec{x} + \vec{y} \in f$, but $c\vec{x}$ is not in

because $0 \geq 0$

S, because c can be negative

$-\infty$

$f: [0, \infty] \longrightarrow [0, \infty]$

Need $f \in V$

~~$f(x) = x^a$ for all $a \in \mathbb{R}$~~

~~$\sin x$, $\cos x$~~ ~~there is no a where $f(x) = 0$~~

~~So zero vector not in set~~

~~$x^a + x^a = 2x^a$~~

Problem 8. (100 points) Define S to be the set of all vectors $\mathbf{x}$ in $\mathcal{R}^3$ whose components x_1, x_2, x_3 satisfy the two restriction equations $x_1 + x_2 = x_3$ and $2x_1 + 5x_2 = x_3$. Prove that S is a subspace of $\mathcal{R}^3$.

Expected: Cite a known theorem or else verify the 3 conditions for the definition of a subspace (see the preceding exam problem).

A

Have that
$$\begin{matrix} x_1 + x_2 = x_3 \\ 2x_1 + 5x_2 = x_3 \end{matrix} \implies x_1 + 4x_2 = 0 \implies \begin{matrix} x_2 = -x_1/4 \\ x_3 = \frac{3x_1}{4} \end{matrix}$$

So the set is all vectors of the form

$$\vec{v} = \begin{bmatrix} t \\ -t/4 \\ 3t/4 \end{bmatrix} \text{ for all } t \in \mathbb{R} \quad = t\left(\begin{matrix}\text{Strang's} \\ \text{Sol}\end{matrix}\right)$$

Let $t = 0$, then

Use Span Theorem: $S = \text{Span}(\text{Strang's Sol})$ is a subspace.

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the zero vector is in S.

Let $\vec{v}$ and $\vec{w}$ be arbitrary vectors in S, then

$$\vec{v} + \vec{w} = \begin{bmatrix} t \\ -t/4 \\ 3t/4 \end{bmatrix} + \begin{bmatrix} s \\ -s/4 \\ 3s/4 \end{bmatrix} = \begin{bmatrix} s+t \\ \frac{-(s+t)}{4} \\ \frac{3(s+t)}{4} \end{bmatrix}, \text{ so } \vec{v} + \vec{w} \in S$$

Let $c \in \mathbb{R}$ and $\vec{v} \in \mathbb{R}$, then

$$c\vec{v} = c\begin{bmatrix} t \\ -t/4 \\ 3t/4 \end{bmatrix} = \begin{bmatrix} ct \\ -(ct)/4 \\ 3(ct)/4 \end{bmatrix} \quad \text{so } c\vec{v} \in S.$$

Therefore, S is a subspace of $\mathbb{R}^n$.

Problem 9. (100 points) Let A be a 4×3 matrix. Assume the columns of $A^T A$ are independent. Prove or disprove that A has independent columns.

Expected: To prove a claim, assemble details and theorem citations to support the claim. To disprove a claim, invent a specific detailed example that violates the claim.

Assume for contradiction that A has dependent columns. Then, there is a nontrivial solution to $A\vec{x} = \vec{0}$.

However,

$$A\vec{x} = \vec{0} \iff A^T A \vec{x} = A^T \vec{0} \iff A^T A \vec{x} = \vec{0}$$

Therefore, $A^T A \vec{x} = \vec{0}$ shares this nontrivial solution.

But this a contradiction, since we assumed the columns of $A^T A$ are independent.

Therefore, A must have independent columns.

Problem 10. (100 points) Let U be a 2×2 matrix with $U^T U = I$. Let $\vec{u}_1, \vec{u}_2$ denote the columns of U. Prove that the columns of U are orthonormal. $\rightarrow u_1 \cdot u_2 = 0 \quad \|u_1\| = 1 \quad \|u_2\| = 1$

A

$$U^T U = I \qquad U = (\vec{u}_1, \vec{u}_2) = \begin{pmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{pmatrix}$$

$$U^T = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

$$U^T U = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \end{pmatrix} = \begin{pmatrix} u_{11}^2 + u_{12}^2 & u_{11}u_{21} + u_{12}u_{22} \\ u_{11}u_{21} + u_{12}u_{22} & u_{21}^2 + u_{22}^2 \end{pmatrix}$$

$$= I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{aligned} u_{11}^2 + u_{12}^2 &= 1 \\ u_{21}^2 + u_{22}^2 &= 1 \\ u_{11}u_{21} + u_{12}u_{22} &= 0 \end{aligned}$$

U orthogonal columns

$$\begin{pmatrix} u_{11} \\ u_{12} \end{pmatrix} \cdot \begin{pmatrix} u_{21} \\ u_{22} \end{pmatrix} = u_{11}u_{21} + u_{12}u_{22} = 0$$ as found above

Columns are orthogonal

as found above

$$\|u_1\| = \sqrt{u_{11}^2 + u_{12}^2} = \sqrt{1} = 1$$

as found above

$$\|u_2\| = \sqrt{u_{21}^2 + u_{22}^2} = \sqrt{1} = 1$$

Length of on

U is orthonormal where $U^T U = I$