MATH 2270-2 Exam 1 Spring 2018

ANSWERS

No books or notes. No electronic devices, please.
Each question has credit 100, with multiple parts given a percentage of the total 100.
If you must write a solution out of order or on the back side, then supply a road map.

Problem 1. (100 points)

Symbol I is used below for the $n \times n$ identity. Notation C^T means the transpose of matrix C. Accept as known theorems the following results:

> **Theorem 1.** If A and B are $n \times n$ and $AB = I$, then $BA = I$.
> **Theorem 2.** If A and B are invertible $n \times n$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.
> **Theorem 3.** If matrices F, G have dimensions allowing FG to be defined, then $(FG)^T = G^TF^T$.
> **Theorem 4.** If matrices F, G have dimensions allowing $F+G$ to be defined, then $(F+G)^T = F^T + G^T$.
> **Theorem 5.** If C is $n \times n$ invertible, then C^T is invertible and $\left(C^T\right)^{-1} = (C^{-1})^T$.
> **Theorem 6.** If A and B are $n \times n$, then $|AB| = |A||B|$ (Determinant Product Theorem).
> **Theorem 7.** Assume A is $n \times n$. Matrix A is invertible if and only $|A| \neq 0$.

In each of parts **(a)** and **(b)**, either invent a counter example or else explain why it is true citing the theorems above.

(a) [50%] If matrices A, B are $n \times n$ with $A^T = A$ and A^{-1} exists, then $A(A^{-1} + B)^T = I + (BA)^T$.

(b) [50%] If matrices A, B are $n \times n$ and A^2B^2 is not invertible, then both A and B have determinant zero.

Answer:
(a) TRUE. Why it is true:

$$\begin{aligned}
A(A^{-1}+B)^T &= A^T(A^{-1}+B)^T \text{ because } A = A^T \\
&= A^T((A^{-1})^T + B^T) \text{ because of Theorem 4.} \\
&= A^T(A^{-1})^T + A^TB^T \text{ by matrix multiply.} \\
&= (A^{-1}A)^T + (BA)^T \text{ because of Theorem 3.} \\
&= I + (BA)^T \text{ because } A \text{ is invertible.}
\end{aligned}$$

(b) FALSE. Let $A = I$ and let B equal the zero matrix. Then A^2B^2 is the zero matrix, which has determinant zero and therefore it is not invertible by Theorem 7. However, $|A| = 1$ even though $|B| = 0$. Conclusion: the hypotheses are true but the conclusion is false.

Problem 2. (100 points)

(1) E_1 represents `combo(1,3,-1)` applied to the identity I.

(2) E_2 represents `swap(2,3)` applied to the identity I.

(3) E_3 represents `mult(2,1/2)` applied to the identity I.

Instead of performing matrix multiplies, we create E with a toolkit sequence as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \quad \texttt{combo(1,3,-1)}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \texttt{swap(2,3)}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \end{pmatrix} \quad \texttt{mult(2,1/2)}$$

Problem 3. (100 points) Let a, b and c denote constants and consider the system of equations

$$\begin{cases} cx + by + z = a \\ (b+c)x - ay + 2z = -a \\ bx + ay + z = -a \end{cases}$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a) [40%] The system has a unique solution for $(b-c)(2a+b) \neq 0$.

(b) [30%] The system has no solution if $b + 2a = 0$ and $a \neq 0$ (don't explain the other possibilities).

(c) [30%] The system has infinitely many solutions if $a = b = c = 0$ (don't explain the other possibilities).

Answer:

The system can be written as

$$\begin{pmatrix} c & b & 1 \\ b+c & -a & 2 \\ b & a & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a \\ -a \\ -a \end{pmatrix}$$

which will be referenced in the solution below as $A\vec{u} = \vec{b}$.

(a) **Uniqueness**: This requires zero free variables. Then the determinant of the coefficient matrix A must be nonzero. After cofactor expansion the determinant is factored as $(b + 2a)(b - c)$. The inverse of the coefficient matrix then exists for $(b + 2a)(b - c) \neq 0$, which implies equation $A\vec{u} = \vec{b}$ has unique solution $\vec{u} = A^{-1}\vec{b}$.

(b) **No solution**: The toolkit of combo, swap and mult are used in part (b). We seek a signal equation when $b + 2a = 0$ and $a \neq 0$. After 3 combo steps the matrix is transformed into

$$A_3 = \begin{pmatrix} c & b & 1 & a \\ -c+b & -2\,b-a & 0 & -3\,a \\ 0 & b+2\,a & 0 & a \end{pmatrix}$$

The last row of A_3 is a signal equation if $b + 2a = 0$ and $a \neq 0$. The combo details are in the Maple code below.

(c) **Infinitely many solutions**: If $a = b = c = 0$, then from part (b)

$$A_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix A_3 has one lead variable and two free variables, because the last two rows of A_3 are zero. A homogeneous problem always has solution zero, therefore it never has a signal equation. Therefore the system has infinitely many solutions.

A full analysis of the three possibilities is fairly complex.

The sequence of steps used in (a), (b), (c) are documented below for maple.

```
combo:=(A,s,t,m)->linalg[addrow](A,s,t,m);
mult:=(A,t,m)->linalg[mulrow](A,t,m);
swap:=(A,s,t)->linalg[swaprow](A,s,t);
A:=(a,b,c)->Matrix([[c,b,1,a],[b+c,-a,2,-a],[b,a,1,-a]]);
linalg[det](A(a,b,c)[1..3,1..3]);
A1:=combo(A(a,b,c),1,2,-2);
A2:=combo(A1,1,3,-1);
A3:=combo(A2,2,3,-1);
```

> **Definition**. Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called **independent** provided solving vector equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ results in the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called **dependent**.

Problem 4. (100 points) Solve parts (a), (b) and (c) using the vectors displayed below. Expected is application of the Pivot Theorem or the definition of independence (above). Details are 75%, answer 25%.

$$\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix},\ \begin{pmatrix}1\\2\\0\\2\\2\end{pmatrix},\ \begin{pmatrix}0\\2\\0\\1\\1\end{pmatrix},\ \begin{pmatrix}1\\0\\0\\2\\2\end{pmatrix}$$

(a) [50%] Show details for the dependence of the 4 vectors.

(b) [20%] List a maximum number of independent vectors extracted from the 4 vectors.

(c) [30%] Explain why the 4 column vectors fail to span the vector space $\mathcal{R}^5$, without using the results from parts (a), (b).

Answer:

(a) The vectors are dependent by the Pivot Theorem because the augmented matrix of the vectors has pivot columns 1,2,4. Therefore, vectors 1, 2, 4 are independent. By the Pivot Theorem, the third vector is a linear combination of the pivot columns 1,2,4.

$$\begin{pmatrix}1&1&0&1\\0&2&2&0\\0&0&0&0\\1&2&1&2\\1&2&1&2\end{pmatrix} \quad \text{has RREF} = \quad \begin{pmatrix}1&0&-1&0\\0&1&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

(b) A maximum number of independent vectors is vectors 1,2,4. The third is dependent upon the others.

(c) Theorem: Less than n vectors in $\mathcal{R}^n$ cannot span $\mathcal{R}^n$.

Problem 5. **(100 points)** Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix}1&0&0&4\\3&0&1&0\\4&0&1&4\\0&0&0&0\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}0\\4\\4\\0\end{pmatrix}$$

Expected: (a) [10%] Augmented matrix, (b) [40%] Toolkit steps for the RREF, (c) [10%] Conversion of RREF to scalar equations, (d) [20%] Last frame Algorithm details to write out the scalar general solution, (e) [20%] Conversion of the scalar general solution to the vector general solution. This answer is in the form of a single vector equation for $\vec{x}$, the solution of system $A\vec{x} = \vec{b}$.

Answer:

The augmented matrix for this system of equations is

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 3 & 0 & 1 & 0 & 4 \\ 4 & 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The reduced row echelon form is found as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 4 & 0 & 1 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{combo(1,2,-3)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 0 & 0 & 1 & -12 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{combo(1,3,-4)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{combo(2,3,-1)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{last frame}$$

The last frame, or RREF, is equivalent to the scalar system

$$\begin{array}{rcrcrcl} x_1 & & & + & 4x_4 & = & 0 \\ & & x_3 & - & 12x_4 & = & 4 \\ & & & & & = & 0 \\ & & & & & = & 0 \end{array}$$

The lead variables are x_1, x_3 and the free variables are x_2, x_4. The last frame algorithm introduces invented symbols t_1, t_2. The free variables are set to these symbols, then back-substitute into the lead variable equations of the last frame to obtain the scalar general solution

$$\begin{aligned} x_1 &= -4t_2, \\ x_2 &= t_1, \\ x_3 &= 4 + 12t_2, \\ x_4 &= t_2. \end{aligned}$$

Strang's *special solutions* are $\vec{v}_1, \vec{v}_2$, obtained as the partial derivatives of $\vec{x}$ on the invented symbols t_1, t_2, respectively. A particular solution $\vec{x}_p$ is obtained by setting all invented symbols to zero. Then

$$\vec{x} = \vec{x}_p + t_1\vec{v}_1 + t_2\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} -4 \\ 0 \\ 12 \\ 1 \end{pmatrix}$$

Problem 6. (100 points) Determinant problem, chapter 3.
Details 75%, answers 25%.

(a) [30%] Assume given 3×3 matrices A, B. Assume E_1, E_2, E_3 are elementary matrices representing respectively a combination, a swap and a multiply by 4. Assume $\det(B) = 5$ and $E_3E_2E_1A = A^2B^3$. Let $C = -2A$. Find all possible values of $\det(C)$.

(b) [30%] Determine all values of x for which A^{-1} exists, where $A = I + C$, I is the 3×3 identity and $C = \begin{pmatrix} 2 & x & -1 \\ x & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

(c) [40%] Let symbols a, b, c denote constants and define

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{pmatrix}$$

Apply the adjugate [adjoint] formula for the inverse

$$A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$$

to find the value of the entry in row 3, column 2 of A^{-1}.

Answer:

(a) Start with the determinant product theorem $|FG| = |F||G|$. Apply it to obtain $|E_3||E_2||E_1||A| = |A|^2|B|^3$. Let $x = |A|$, $|B| = 5$, $|E_1| = 1$, $|E_2| = -1$ and $|E_3| = 4$ in this equation to obtain quadratic equation $(4)(-1)(1)x = x^2(4)^3$. Then solve for $x = 0$ or $x = -4/4^3$. Then $|C| = |(-2I)A| = |-2I||A| = -8x$. The answer is $|C| = -8x = 0$ or $|C| = -8x = (-8)(-4/4^3) = \frac{1}{2}$.

(b) Find $C + I = \begin{pmatrix} 3 & x & -1 \\ x & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, then evaluate its determinant, to eventually solve for

$x = -2$. Used here is F^{-1} exists if and only if $|F| \neq 0$. The answer is $I + C$ has an inverse for all $x \neq -2$.

(c) Find the cross-out determinant in row 2, column 3 (no mistake, the transpose swaps rows and columns). Form the fraction, top=checkboard sign times cross-out determinant, bottom=$|A|$. The value is $\frac{c}{2} - b$. A maple check:

```
C4:=Matrix([[1,0,0,0],[1,-2,0,0],[a,b,0,1],[1,c,1,2]]);
1/C4;  The inverse matrix
C5:=linalg[minor](C4,2,3);
top:=linalg[det](C5)*(-1)^(2+3);bot:=linalg[det](C4);top/bot;
# ans =c-2b divided by 2
```

End Exam 1.