Chapter 3

Linear Algebraic Equations

Contents

This introduction to linear algebraic equations requires only a college algebra background. **Vector and matrix notation is not used**. The subject of linear algebra, using vectors, matrices and related tools, appears later in the text; see Chapter 5.

The topics studied are *linear equations*, *general solution*, *reduced echelon system*, *basis*, *nullity*, *rank* and *nullspace*. Introduced here are the **three possibilities**, the **toolkit sequence**, which uses the three rules **swap**, **combination** and **multiply**, and finally the method of **elimination**, in literature called **Gauss-Jordan elimination** or **Gaussian elimination**

3.1 Linear Systems of Equations

Background from college algebra includes systems of linear algebraic equations like

$$(1) \qquad \left\{ \begin{array}{rcrcl} 3x & + & 2y & = & 1, \\ x & - & y & = & 2. \end{array} \right.$$

A **solution** (x, y) of **non-homogeneous system** (1) is a pair of values that simultaneously satisfy both equations. This example has unique solution $x = 1$, $y = -1$.

The **homogeneous system** corresponding to (1) is an auxiliary system invented by replacing the right sides of the equations by zero and symbols

x, y by new symbols u, v:

$$(2) \quad \begin{cases} 3u + 2v = 0, \\ u - v = 0. \end{cases}$$

The reader should pause to verify that system (2) has unique solution $u = 0$, $v = 0$.

It is unexpected, in general, that the original system (solution $x = 1, y = -1$) has any solutions in common with the invented homogeneous system (solution $u = 0, v = 0$). Theory provides **superposition** to relate the solutions of the two systems.

Unique solutions have emphasis in college algebra courses. In this chapter we study in depth the cases for **no solution** and **infinitely many solutions**. These two cases are illustrated by the examples

No Solution

$$(3) \quad \begin{cases} x - y = 0, \\ 0 = 1. \end{cases}$$

Infinitely Many Solutions

$$(4) \quad \begin{cases} x - y = 0, \\ 0 = 0. \end{cases}$$

Equations (3) cannot have a solution because of the **signal equation** $0 = 1$, a false equation. Equations (4) have one solution (x, y) for each point on the 45° line $x - y = 0$, therefore system (4) has infinitely many solutions.

The Three Possibilities

Solutions of general linear systems with m equations in n unknowns may be classified into exactly **three possibilities**:

1. No solution.
2. Infinitely many solutions.
3. A unique solution.

General Linear Systems

Given numbers $a_{11}, \ldots, a_{mn}, b_1, \ldots, b_m$, a **nonhomogeneous system** of m linear equations in n **unknowns** $x_1, x_2, \ldots, x_n$ is the system

$$(5) \quad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Constants $a_{11}, \ldots, a_{mn}$ are called the **coefficients** of system (5). Constants $b_1, \ldots, b_m$ are collectively referenced as the **right hand side**, **right side** or **RHS**.

The associated **homogeneous system** corresponding to system (5) is **invented** by replacing the right side by zero:

$$
\begin{array}{rcl}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & 0, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & 0, \\
 & \vdots & \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & 0.
\end{array} \tag{6}
$$

Convention dictates using the same variable list $x_1, \ldots, x_n$. This abuse of notation impacts casual readers: see example systems (1) and (2).

An assignment of possible values $x_1, \ldots, x_n$ which simultaneously satisfy all equations in (5) is called a **solution** of system (5). **Solving** system (5) refers to the process of finding all possible solutions of (5). The system (5) is called **consistent** if it has a solution and otherwise it is called **inconsistent**.

The Toolkit of Three Rules

Two systems (5) are said to be **equivalent** provided they have exactly the same solutions. For the purpose of solving systems, there is a toolkit of three reversible operations on equations which can be applied to obtain equivalent systems. These rules *neither create nor destroy solutions* of the original system:

Table 1. The Three Rules

Swap	Two equations can be interchanged without changing the solution set.
Multiply	An equation can be multiplied by $m \neq 0$ without changing the solution set.
Combination	A multiple of one equation can be added to a different equation without changing the solution set.

The last two rules replace an existing equation by a new one. A **swap** repeated reverses the swap operation. A **multiply** is reversed by multiplication by $1/m$, whereas the **combination** rule is reversed by subtracting the equation–multiple previously added. In short, the three operations are **reversible**.

Theorem 1 (Equivalent Systems)
A second system of linear equations, obtained from the first system of linear equations by a finite number of toolkit operations, has exactly the same solutions as the first system.

Exposition. Writing a set of equations and its equivalent system under toolkit rules demands that all equations be copied, not just the affected equation(s). Generally, each displayed system changes just one equation, the single exception being a swap of two equations. Within an equation, variables appear left-to-right in variable list order. Equations that contain no variables, typically $0 = 0$, are displayed last.

Documenting the three rules. In blackboard and hand-written work, the acronyms **swap**, **mult** and **combo**, replace the longer terms *swap*, *multiply* and *combination*. They are placed next to the first changed equation. In cases where precision is required, additional information is supplied, namely the **source** and **target** equation numbers s, t and the multiplier $m \neq 0$ or c. Details:

Table 2. Documenting toolkit operations with `swap, mult, combo`**.**

`swap(s,t)`	Swap equations s and t.
`mult(t,m)`	Multiply target equation t by multiplier $m \neq 0$.
`combo(s,t,c)`	Multiply source equation s by multiplier c and add to target equation t.

The acronyms in Table 2 match usage in the computer algebra system `maple`, for package `linalg` and functions `swaprow`, `mulrow` and `addrow`.

Inverses of the Three Rules. Each toolkit operation `swap, mult, combo` has an inverse, which is documented in the following table. The facts can be used to back up several steps, unearthing a previous step to which a sequence of toolkit operations were performed.

Table 3. Inverses of toolkit operations `swap, mult, combo`**.**

Operation	Inverse
`swap(s,t)`	`swap(s,t)`
`mult(t,m)`	`mult(t,1/m)`
`combo(s,t,c)`	`combo(s,t,-c)`

To illustrate, suppose `swap(1,3), combo(1,2,-3), mult(2,4)` are used to obtain the current linear equations. Then the linear system three steps back can be obtained from the current system by applying the inverse steps in reverse order: `mult(2,1/4), combo(1,2,3), swap(1,3)`.

Solving Equations with Geometry

In the plane ($n = 2$) and in 3-space ($n = 3$), equations (5) have a geometric interpretation that can provide valuable intuition about possible solutions. College algebra courses might have omitted the case of *no solutions* or *infinitely many solutions*, discussing only the case of a single unique solution. In contrast, all cases are considered here.

Plane Geometry. A **straight line** may be represented as an equation $Ax + By = C$. Solving the system

$$
\begin{array}{rcrcl}
a_{11}x & + & a_{12}y & = & b_1 \\
a_{21}x & + & a_{22}y & = & b_2
\end{array}
\tag{7}
$$

is the geometrical equivalent of finding all possible (x, y)-intersections of the lines represented in system (7). The distinct geometrical possibilities appear in Figures 1, 2 and 3.

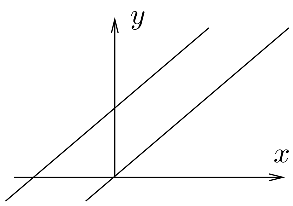

Figure 1. Parallel lines, no solution.

$$
\begin{array}{rcl}
-x + y & = & 1, \\
-x + y & = & 0.
\end{array}
$$

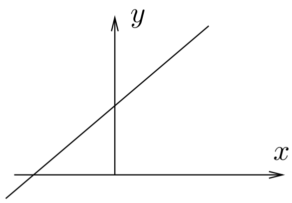

Figure 2. Identical lines, infinitely many solutions.

$$
\begin{array}{rcl}
-x + y & = & 1, \\
-2x + 2y & = & 2.
\end{array}
$$

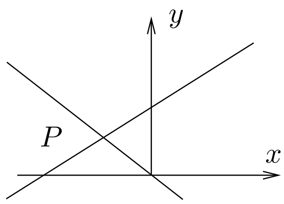

Figure 3. Non-parallel distinct lines, one solution at the unique intersection point P.

$$
\begin{array}{rcl}
-x + y & = & 2, \\
x + y & = & 0.
\end{array}
$$

Space Geometry. A **plane** in xyz-space is given by an equation $Ax + By + Cz = D$. The vector $A\vec{\imath} + B\vec{\jmath} + C\vec{k}$ is **normal** to the plane. An equivalent equation is $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$, where (x_0, y_0, z_0) is a given point in the plane. Solving system

$$
\begin{array}{rcrcrcl}
a_{11}x & + & a_{12}y & + & a_{13}z & = & b_1 \\
a_{21}x & + & a_{22}y & + & a_{23}z & = & b_2 \\
a_{31}x & + & a_{32}y & + & a_{33}z & = & b_3
\end{array}
\tag{8}
$$

is the geometric equivalent of finding all possible (x, y, z)-intersections of the planes represented by system (8). Illustrated in Figures 4–11 are some interesting geometrical possibilities.

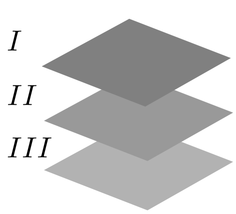

Figure 4. Three Parallel Shelves. Planes I, II, III are parallel. There is no intersection point.

$$I : z = 2, \quad II : z = 1, \quad III : z = 0.$$

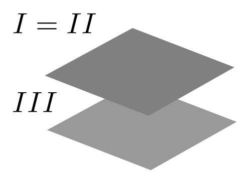

Figure 5. Two Parallel Shelves. Planes I, II are equal and parallel to plane III. There is no intersection point.

$$I: 2z = 2, \quad II: z = 1, \quad III: z = 0.$$

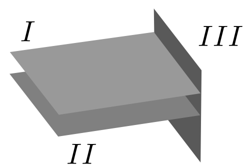

Figure 6. Book shelf. Two planes I, II are distinct and parallel. There is no intersection point.

$$I: z = 2, \quad II: z = 1, \quad III: y = 0.$$

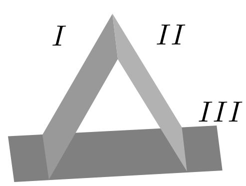

Figure 7. Pup tent. Two non-parallel planes I, II meet in a line which never meets plane III. There are no intersection points.

$$I: y+z = 0, \quad II: y-z = 0, \quad III: z = -1.$$

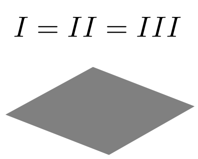

Figure 8. Three Identical Shelves. Planes I, II, III are equal. There are infinitely many intersection points.

$$I: z = 1, \quad II: 2z = 2, \quad III: 3z = 3.$$

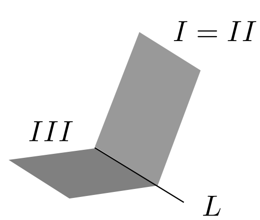

Figure 9. Open book. Equal planes I, II meet another plane III in a line L. There are infinitely many intersection points.

$$I: y + z = 0, \; II: 2y + 2z = 0, \; III: z = 0.$$

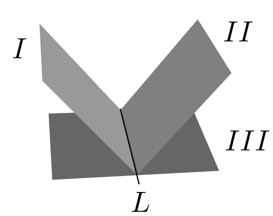

Figure 10. Saw Tooth. Two non-parallel planes I, II meet in a line L which lies in a third plane III. There are infinitely many intersection points.

$$I: -y+z = 0, \quad II: y+z = 0, \quad III: z = 0.$$

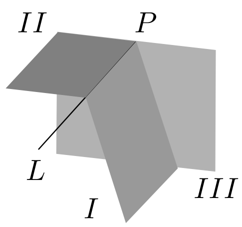

Figure 11. Knife Cuts an Open Book. Two non-parallel planes I, II meet in a line L not parallel to plane III. There is a *unique point* P of intersection of all three planes.

$$I : y + z = 0, \quad II : z = 0, \quad III : x = 0.$$

Examples and Methods

1 Example (Toolkit) Given system $\left|\begin{array}{ccccccc} x & & & + & 4z & = & 1 \\ x & + & y & + & 4z & = & 3 \\ & & & & z & = & 2 \end{array}\right|$, find the system that results from `swap(1,2)` followed by `combo(2,1,-1)`.

Solution: The steps are as follows, with the equivalent system equal to the last display.

$\left|\begin{array}{ccccccc} x & & & + & 4z & = & 1 \\ x & + & y & + & 4z & = & 3 \\ & & & & z & = & 2 \end{array}\right|$ Original system.

$\left|\begin{array}{ccccccc} x & + & y & + & 4z & = & 3 \\ x & & & + & 4z & = & 1 \\ & & & & z & = & 2 \end{array}\right|$ `swap(1,2)`

$\left|\begin{array}{ccccccc} & & y & & & = & 2 \\ x & & & + & 4z & = & 1 \\ & & & & z & = & 2 \end{array}\right|$ `combo(2,1,-1)`

Calculations for `combo(2,1,-1)` can be done on scratch paper. Experts do the arithmetic column-by-column, using no scratch paper. Here's the details for the scratch paper arithmetic:

$$\begin{array}{ccccccccl} 1x & + & 0y & + & 4z & = & 1 & \text{Equation 2} \\ 1x & + & 1y & + & 4z & = & 3 & \text{Equation 1} \end{array}$$

$$\begin{array}{ccccccccl} -1x & + & 0y & - & 4z & = & -1 & \text{Equation 2 times -1} \\ 1x & + & 1y & + & 4z & = & 3 & \text{Equation 1} \end{array}$$

Add on the columns, replacing the second equation.

$$\begin{array}{ccccccccl} -1x & + & 0y & - & 4z & = & -1 & \text{Equation 2 times -1} \\ 0x & + & 1y & + & 0z & = & 2 & \text{Equation 1 + (-1)(Equation 2)} \end{array}$$

The last equation replaces equation 1 and the label `combo(2,1,-1)` is written next to the replacement. All of the scratch work is discarded.

2 Example (Inverse Toolkit) Let system $\left|\begin{array}{ccccccc} x & & & - & 3z & = & -1 \\ & & 2y & + & 6z & = & 4 \\ & & & & z & = & 3 \end{array}\right|$ be produced by toolkit operations `mult(2,2)` and `combo(2,1,-1)`. Find the original system.

Solution: We begin by writing the given toolkit operation inverses, in reverse order, as `combo(2,1,1)` and `mult(2,1/2)`. The operations, in this order, are performed on the given system, to find the original system two steps back, in the last display.

$$\left|\begin{array}{ccccccc} x & & & - & 3z & = & -1 \\ & & 2y & + & 6z & = & 4 \\ & & & & z & = & 3 \end{array}\right|$$ Given system.

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 3 \\ & & 2y & + & 6z & = & 4 \\ & & & & z & = & 3 \end{array}\right|$$ `combo(2,1,1)`
One step back.

$$\left|\begin{array}{ccccccc} x & + & 2y & + & 3z & = & 3 \\ & & y & + & 3z & = & 2 \\ & & & & z & = & 3 \end{array}\right|$$ `mult(2,1/2)`
Two steps back.

3 Example (Planar System) Classify the system geometrically as one of the three types displayed in Figures 1, 2, 3. Then solve for x and y.

$$\left|\begin{array}{ccccc} x & + & 2y & = & 1, \\ 3x & + & 6y & = & 3. \end{array}\right| \quad (9)$$

Solution: The second equation, divided by 3, gives the first equation. In short, the two equations are proportional. The lines are geometrically **equal lines**, as in Figure 2. The two equations are equivalent to the system

$$\left|\begin{array}{ccccc} x & + & 2y & = & 1, \\ & & 0 & = & 0. \end{array}\right|$$

To solve the system means to find all points (x, y) simultaneously common to both lines, which are all points (x, y) on $x + 2y = 1$.

A parametric representation of this line is possible, obtained by setting $y = t$ and then solving for $x = 1 - 2t$, $-\infty < t < \infty$. We report the solution as a parametric solution, but the first solution is also valid.

$$\begin{array}{l} x = 1 - 2t, \\ y = t. \end{array}$$

4 Example (No Solution) Classify the system geometrically as the type displayed in Figure 1. Explain why there is no solution.

$$\left|\begin{array}{ccccc} x & + & 2y & = & 1, \\ 3x & + & 6y & = & 6. \end{array}\right| \quad (10)$$

Solution: The second equation, divided by 3, gives $x + 2y = 2$, a line parallel to the first line $x + 2y = 1$. The lines are geometrically **parallel lines**, as in Figure 1. The two equations are equivalent to the system

$$\left|\begin{array}{ccccc} x & + & 2y & = & 1, \\ x & + & 2y & = & 2. \end{array}\right|$$

To solve the system means to find all points (x, y) simultaneously common to both lines, which are all points (x, y) on $x + 2y = 1$ and also on $x + 2y = 2$. If such a point (x, y) exists, then $1 = x + 2y = 2$ or $1 = 2$, a contradictory **signal equation**. Because $1 = 2$ is **false**, then no common point (x, y) exists and we report **no solution**.

Some readers will want to continue and write equations for x and y, a *solution* to the problem. We emphasize that this is not possible, because there is no solution at all.

The presence of a signal equation, which is a false equation used primarily to detect no solution, will appear always in the solution process for a system of equations that has no solution. Generally, this signal equation, if present, will be distilled to the single equation "$0 = 1$." For instance, $0 = 2$ can be distilled to $0 = 1$ by dividing the first signal equation by 2.

Exercises 3.1

Toolkit. Compute the equivalent system of equations.

1. Given $\left|\begin{array}{rcrcrcl} x & & & + & 2z & = & 1 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$, find the system that results from `combo(2,1,-1)`.

2. Given $\left|\begin{array}{rcrcrcl} x & & & + & 2z & = & 1 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$, find the system that results from `swap(1,2)` followed by `combo(2,1,-1)`.

3. Given $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$, find the system that results from `combo(1,2,-1)`.

4. Given $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$, find the system that results from `swap(1,2)` followed by `combo(1,2,-1)`.

5. Given $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$, find the system that results from `swap(2,3)`, `combo(2,1,-1)`.

6. Given $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$, find the system that results from `mult(2,1/3)`, `combo(1,2,-1)`, `swap(2,3)`, `swap(1,2)`.

Inverse Toolkit. Compute the equivalent system of equations.

7. If $\left|\begin{array}{rcrcrcr} & - & y & & & = & -3 \\ x & + & y & + & 2z & = & 4 \\ & & & & z & = & 0 \end{array}\right|$ resulted from `combo(2,1,-1)`, then find the original system.

8. If $\left|\begin{array}{rcrcrcl} & & y & & & = & 3 \\ x & & & + & 2z & = & 1 \\ & & & & z & = & 0 \end{array}\right|$ resulted from `swap(1,2)` followed by `combo(2,1,-1)`, then find the original system.

9. If $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ & & y & - & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$ resulted from `combo(1,2,-1)`, then find the original system.

10. If $\left|\begin{array}{rcrcrcl} x & & & + & 3z & = & 1 \\ x & + & y & + & 3z & = & 4 \\ & & & & z & = & 1 \end{array}\right|$ resulted from `swap(1,2)` followed by `combo(2,1,2)`, then find the original system.

11. If $\left|\begin{array}{rcrcl} y & + & z & = & 2 \\ 3y & + & 3z & = & 6 \\ y & & & = & 0 \end{array}\right|$ resulted from `mult(2,-1)`, `swap(2,3)`, `combo(2,1,-1)`, then find the original system.

12. If $\left|\begin{array}{rcl} 2y + z &=& 2 \\ 3y + 3z &=& 6 \\ y &=& 0 \end{array}\right|$ resulted from `mult(2,1/3)`, `combo(1,2,-1)`, `swap(2,3)`, `swap(1,2)`, then find the original system.

Planar System. Solve the xy–system and interpret the solution geometrically as

(a) **parallel lines**

(b) **equal lines**

(c) **intersecting lines**.

13. $\left|\begin{array}{rcl} x + y &=& 1, \\ y &=& 1 \end{array}\right|$

14. $\left|\begin{array}{rcl} x + y &=& -1 \\ x &=& 3 \end{array}\right|$

15. $\left|\begin{array}{rcl} x + y &=& 1 \\ x + 2y &=& 2 \end{array}\right|$

16. $\left|\begin{array}{rcl} x + y &=& 1 \\ x + 2y &=& 3 \end{array}\right|$

17. $\left|\begin{array}{rcl} x + y &=& 1 \\ 2x + 2y &=& 2 \end{array}\right|$

18. $\left|\begin{array}{rcl} 2x + y &=& 1 \\ 6x + 3y &=& 3 \end{array}\right|$

19. $\left|\begin{array}{rcl} x - y &=& 1 \\ -x - y &=& -1 \end{array}\right|$

20. $\left|\begin{array}{rcl} 2x - y &=& 1 \\ x - 0.5y &=& 0.5 \end{array}\right|$

21. $\left|\begin{array}{rcl} x + y &=& 1 \\ x + y &=& 2 \end{array}\right|$

22. $\left|\begin{array}{rcl} x - y &=& 1 \\ x - y &=& 0 \end{array}\right|$

System in Space. For each xyz–system:

(a) If no solution, then report **three identical shelves**, **two parallel shelves**, **pup tent** or **book shelf**.

(b) If infinitely many solutions, then report **one shelf**, **open book** or **saw tooth**.

(c) If a unique intersection point, then report the values of x, y and z.

23. $\left|\begin{array}{rcl} x - y + z &=& 2 \\ x &=& 1 \\ y &=& 0 \end{array}\right|$

24. $\left|\begin{array}{rcl} x + y - 2z &=& 3 \\ x &=& 2 \\ z &=& 1 \end{array}\right|$

25. $\left|\begin{array}{rcl} x - y &=& 2 \\ x - y &=& 1 \\ x - y &=& 0 \end{array}\right|$

26. $\left|\begin{array}{rcl} x + y &=& 3 \\ x + y &=& 2 \\ x + y &=& 1 \end{array}\right|$

27. $\left|\begin{array}{rcl} x + y + z &=& 3 \\ x + y + z &=& 2 \\ x + y + z &=& 1 \end{array}\right|$

28. $\left|\begin{array}{rcl} x + y + 2z &=& 2 \\ x + y + 2z &=& 1 \\ x + y + 2z &=& 0 \end{array}\right|$

29. $\left|\begin{array}{rcl} x - y + z &=& 2 \\ 2x - 2y + 2z &=& 4 \\ y &=& 0 \end{array}\right|$

30. $\left|\begin{array}{rcl} x + y - 2z &=& 3 \\ 3x + 3y - 6z &=& 6 \\ z &=& 1 \end{array}\right|$

31. $\left|\begin{array}{rcl} x - y + z &=& 2 \\ 0 &=& 0 \\ 0 &=& 0 \end{array}\right|$

32. $\left|\begin{array}{rcl} x + y - 2z &=& 3 \\ 0 &=& 0 \\ 1 &=& 1 \end{array}\right|$

33. $\left|\begin{array}{rcl} x + y &=& 2 \\ x - y &=& 2 \\ y &=& -1 \end{array}\right|$

34. $\begin{vmatrix} x & & - & 2z & = & 4 \\ x & & + & 2z & = & 0 \\ & & & z & = & 2 \end{vmatrix}$

35. $\begin{vmatrix} & y & + & z & = & 2 \\ & 3y & + & 3z & = & 6 \\ & y & & & = & 0 \end{vmatrix}$

36. $\begin{vmatrix} x & & + & 2z & = & 1 \\ 4x & & + & 8z & = & 4 \\ & & & z & = & 0 \end{vmatrix}$

3.2 Filmstrips and Toolkit Sequences

Expert on Video. A linear algebra expert solves a linear system of equations by hand. A video documents the details, starting with the original system of equations and ending with the solution of the linear system. At each application of a toolkit operation `swap`, `combo` or `mult`, the system of equations is re-written.

Filmstrip. The documentary video is edited into an ordered sequence of images, a **filmstrip** which eliminates all arithmetic details. The cropped images are the selected frames which record the result of each computation: only major toolkit steps appear (see Table 4).

Table 4. A Toolkit Sequence.
Each image is a cropped frame from a filmstrip, obtained by editing a documentary video of a person solving the linear system.

Frame 1	Frame 2	Frame 3
Original System $\begin{cases} x - y = 2, \\ 3y = -3. \end{cases}$	Apply `mult(2,1/3)` $\begin{cases} x - y = 2, \\ y = -1. \end{cases}$	Apply `combo(2,1,1)` $\begin{cases} x = 1, \\ y = -1. \end{cases}$

Definition 1 (Toolkit Sequence)
A sequence of select, cropped filmstrip images from a documentary video of a person solving a linear system is called a **toolkit sequence**, provided the images are devoid of arithmetic detail and each toolkit operation `swap`, `combo` or `mult` is fully documented.

Lead Variables

A variable chosen from the variable list x, y is called a **lead variable** provided it appears just once in the entire system of equations, and in addition, its appearance reading left-to-right is first, with coefficient one. The same definition applies to arbitrary variable lists $x_1, x_2, \ldots, x_n$.

Illustration. Symbol x is a lead variable in all three frames of the toolkit sequence in Table 4. But symbol y fails to be a lead variable in toolkit frames 1 and 2. In the final frame, both x and y are lead variables.

A **free variable** is a non-lead variable, detectable only from a toolkit frame in which every non-zero equation has a lead variable.

A consistent system in which every variable is a lead variable must have a unique solution. The system must look like the final frame of the toolkit sequence in Table 4. More precisely, the variables appear in variable list order to the left of the equal sign, each variable appearing just once, with numbers to the right of the equal sign.

Unique Solution

To solve a system with a unique solution, we apply the toolkit operations of swap, multiply and combination (acronyms `swap, mult, combo`), one operation per frame, until the last frame displays the unique solution.

Because all variables will be lead variables in the last toolkit frame, we seek to create a new lead variable in each frame. Sometimes, this is not possible, even if it is the general objective. Exceptions are swap and multiply operations, which are often used to prepare for creation of a lead variable. Listed in Table 5 are the rules and conventions that we use to create toolkit sequences.

Table 5. Conventions and rules for toolkit sequence creation.

Order of Variables. Variables in equations appear in variable list order to the left of the equal sign.

Order of Equations. Equations are listed in variable list order inherited from their lead variables. Equations without lead variables appear next. Equations without variables appear last. Multiple swap operations convert any system to this convention.

New Lead Variable. Select a new lead variable as the *first variable*, in variable list order, which appears among the equations without a lead variable.

An illustration:

$$\begin{array}{rcrcrcl} & & y & + & 4z & = & 2, \\ x & + & y & & & = & 3, \\ x & + & 2y & + & 3z & = & 4. \end{array}$$

Frame 1. Original system.

$$\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 4, \\ x &+& y & & &=& 3, \\ & & y &+& 4z &=& 2. \end{array}$$

Frame 2.

`swap(1,3)`

$$\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 4, \\ &-& y &-& 3z &=& -1, \\ & & y &+& 4z &=& 2. \end{array}$$

Frame 3.
`combo(1,2,-1)`

$$\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 4, \\ &-& y &-& 3z &=& -1, \\ & & & & z &=& 1. \end{array}$$

Frame 4.

`combo(2,3,1)`

$$\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 4, \\ & & y &+& 3z &=& 1, \\ & & & & z &=& 1. \end{array}$$

Frame 5.
`mult(2,-1)`

$$\begin{array}{rcrcrcr} x & & &-& 3z &=& 2, \\ & & y &+& 3z &=& 1, \\ & & & & z &=& 1. \end{array}$$

Frame 6.
`combo(2,1,-2)`

$$\begin{array}{rcrcrcr} x & & &-& 3z &=& 2, \\ & & y & & &=& -2, \\ & & & & z &=& 1. \end{array}$$

Frame 7.
`combo(3,2,-3)`

$$\begin{array}{rcrcrcr} x & & & & &=& 5, \\ & & y & & &=& -2, \\ & & & & z &=& 1. \end{array}$$

Frame 8. `combo(3,1,3)`
Last Frame.
Unique solution.

No Solution

A special case occurs in a toolkit sequence, when a nonzero equation occurs having no variables. Called a **signal equation**, its occurrence signals **no solution**, because the equation is false. Normally, we halt the toolkit sequence at the point of first discovery, and then declare no solution. An illustration:

$$\begin{array}{rcrcrcr} & & y &+& 3z &=& 2, \\ x &+& y & & &=& 3, \\ x &+& 2y &+& 3z &=& 4. \end{array}$$

Frame 1. Original system.

$$\begin{array}{rcrcrcr} x &+& 2y &+& 3z &=& 4, \\ x &+& y & & &=& 3, \\ & & y &+& 3z &=& 2. \end{array}$$

Frame 2.

`swap(1,3)`

$$\begin{array}{|rcrcrcr|}\hline x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & y & + & 3z & = & 2. \\ \hline\end{array}$$

Frame 3.
`combo(1,2,-1)`

$$\begin{array}{|rcrcrcr|}\hline x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & & & 0 & = & 1. \\ \hline\end{array}$$

Frame 4.
Signal Equation $0 = 1$.
`combo(2,3,1)`

The signal equation $\boxed{0 = 1}$ is a false equation, therefore the last toolkit frame has no solution. Because the toolkit neither creates nor destroys solutions, then the original system in the first frame has **no solution**.

Readers who want to go on and write an answer for the system must be warned that **no such possibility exists**. Values cannot be assigned to any variables in the case of no solution. This can be perplexing, especially in a final frame like

$$\begin{array}{|rcr|}\hline x & = & 4, \\ z & = & -1, \\ 0 & = & 1. \\ \hline\end{array}$$

While it is true that x and z were assigned values, the final signal equation $0 = 1$ is false, meaning any answer is impossible. There is no possibility to write equations for all variables. There is **no solution**. It is a **tragic error** to claim $x = 4$, $z = -1$ is a solution.

Infinitely Many Solutions

A system of equations having infinitely many solutions is solved from a toolkit sequence construction that parallels the unique solution case. The same quest for lead variables is made, hoping in the final frame to have just the variable list on the left and numbers on the right.

The stopping criterion which identifies the final frame, in either the case of a unique solution or infinitely many solutions, is exactly the same:

> **Last Frame Test**. A toolkit step is the **last frame** when every nonzero equation has a lead variable. Remaining equations have the form $0 = 0$.

Any variables that are not lead variables, in the final frame, are called **free variables**, because their values are completely undetermined. Any **missing variable** must be a free variable.

$$\begin{array}{|rcrcrcr|}\hline & & y & + & 3z & = & 1, \\ x & + & y & & & = & 3, \\ x & + & 2y & + & 3z & = & 4. \\ \hline\end{array}$$

Frame 1. Original system.

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ x & + & y & & & = & 3, \\ & & y & + & 3z & = & 1. \end{array}$$

Frame 2.

`swap(1,3)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & y & + & 3z & = & 1. \end{array}$$

Frame 3.
`combo(1,2,-1)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & - & y & - & 3z & = & -1, \\ & & & & 0 & = & 0. \end{array}$$

Frame 4.

`combo(2,3,1)`

$$\begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4, \\ & & y & + & 3z & = & 1, \\ & & & & 0 & = & 0. \end{array}$$

Frame 5.
`mult(2,-1)`

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & 2, \\ & & y & + & 3z & = & 1, \\ & & & & 0 & = & 0. \end{array}$$

Frame 6. `combo(2,1,-2)`
Last Frame.
Lead=x, y, Free=z.

Last Toolkit Frame to General Solution

Once the *last frame* of the toolkit sequence is obtained, then the general solution can be written by a fixed and easy-to-learn algorithm.

Last Frame Algorithm

This process applies only to the last toolkit step in the case of infinitely many solutions.

(1) **Assign invented symbols** t_1, t_2, ... to the free variables.
(2) **Isolate** each lead variable.
(3) **Back-substitute** the free variable invented symbols.

To illustrate, assume the last toolkit step of the sequence is

$$\begin{array}{rcrcrcr} x & & & - & 3z & = & 2, \\ & & y & + & 3z & = & 1, \\ & & & & 0 & = & 0, \end{array}$$

Last Frame.
Lead variables x, y.

then the general solution is written as follows.

$$z = t_1$$

The free variable z is assigned symbol t_1.

$$\begin{array}{l} x = 2 + 3z, \\ y = 1 - 3z \end{array}$$

The lead variables are x, y. Isolate them left.

$x = 2 + 3t_1,$ $y = 1 - 3t_1,$ $z = t_1.$	Back-substitute. Solution found.

In the **last frame**, variables appear left of the equal sign in variable list order. Only invented symbols[1] appear right of the equal sign. The expression is called a **standard general solution**. The meaning:

Nothing Skipped	Each solution of the system of equations can be obtained by specializing the invented symbols t_1, t_2, ... to particular numbers.
It Works	The general solution expression satisfies the system of equations for all possible values of the symbols t_1, t_2, ...

General Solution and the Last Frame Algorithm

An additional illustration will be given for the last frame algorithm. Assume **variable list order** x, y, z, w, u, v for the **last frame**

$$\begin{array}{rl} \boxed{x} + z + u + v &= 1, \\ \boxed{y} - u + v &= 2, \\ \boxed{w} + 2u - v &= 0. \end{array} \tag{11}$$

Every nonzero equation above has a lead variable. The **lead variables** in (11) are the boxed symbols x, y, w. The **free variables** are z, u, v.

Assign invented symbols t_1, t_2, t_3 to the free variables and back-substitute in (11) to obtain a **standard general solution**

$$\left\{\begin{array}{lcl} x &=& 1 - t_1 - t_2 - t_3, \\ y &=& 2 + t_2 - t_3, \\ w &=& -2t_2 + t_3, \\ z &=& t_1, \\ u &=& t_2, \\ v &=& t_3. \end{array}\right. \quad \text{or} \quad \left\{\begin{array}{lcl} x &=& 1 - t_1 - t_2 - t_3, \\ y &=& 2 + t_2 - t_3, \\ z &=& t_1, \\ w &=& -2t_2 + t_3, \\ u &=& t_2, \\ v &=& t_3. \end{array}\right.$$

It is demanded by convention that general solutions be displayed in variable list order. This is why the above display bothers to re-write the equations in the new order on the right.

Exercises 3.2

Lead and free variables. For each system assume variable list $x_1, \ldots, x_5$.

List the lead and free variables.

[1]Computer algebra system `maple` uses invented symbols t_1, t_2, t_3, ... and we follow the convention.

1. $\left|\begin{array}{l} x_2+3x_3 \quad =0 \\ \qquad\quad x_4 \; =0 \\ \qquad\qquad 0=0 \end{array}\right|$

2. $\left|\begin{array}{l} x_2 \qquad\qquad\quad =0 \\ \quad x_3 \quad +3x_5=0 \\ \qquad x_4+2x_5=0 \end{array}\right|$

3. $\left|\begin{array}{l} x_2+3x_3 \quad =0 \\ \qquad\quad x_4 \; =0 \\ \qquad\qquad 0=0 \end{array}\right|$

4. $\left|\begin{array}{l} x_1+2x_2+3x_3 \quad =0 \\ \qquad\qquad\quad x_4 \; =0 \\ \qquad\qquad\qquad 0=0 \end{array}\right|$

5. $\left|\begin{array}{l} x_1+2x_2+3x_3 =0 \\ \qquad\qquad\qquad 0=0 \\ \qquad\qquad\qquad 0=0 \\ \qquad\qquad\qquad 0=0 \end{array}\right|$

6. $\left|\begin{array}{l} x_1+x_2 \quad =0 \\ \qquad x_3 \; =0 \\ \qquad\quad 0=0 \end{array}\right|$

7. $\left|\begin{array}{l} x_1+x_2+3x_3+5x_4 \quad =0 \\ \qquad\qquad\qquad\quad x_5=0 \\ \qquad\qquad\qquad\quad 0=0 \end{array}\right|$

8. $\left|\begin{array}{l} x_1+2x_2 \quad +3x_4+4x_5=0 \\ \qquad\qquad x_3+\ x_4+\ x_5=0 \\ \qquad\qquad\qquad\qquad 0=0 \end{array}\right|$

9. $\left|\begin{array}{l} x_3+2x_4 \quad =0 \\ \qquad\quad x_5=0 \\ \qquad\quad 0=0 \\ \qquad\quad 0=0 \end{array}\right|$

10. $\left|\begin{array}{l} x_4+x_5=0 \\ \qquad 0=0 \\ \qquad 0=0 \\ \qquad 0=0 \end{array}\right|$

11. $\left|\begin{array}{l} x_2 \quad +5x_4 \quad =0 \\ \quad x_3+2x_4 \quad =0 \\ \qquad\qquad x_5=0 \\ \qquad\qquad 0=0 \end{array}\right|$

12. $\left|\begin{array}{l} x_1 \quad +3x_3 \qquad\quad =0 \\ \quad x_2 \qquad +x_4 \quad =0 \\ \qquad\qquad\qquad x_5=0 \\ \qquad\qquad\qquad 0=0 \end{array}\right|$

Elementary Operations. Consider the 3×3 system

$$\begin{array}{rcrcrcl} x &+& 2y &+& 3z &=& 2, \\ -2x &+& 3y &+& 4z &=& 0, \\ -3x &+& 5y &+& 7z &=& 3. \end{array}$$

Define symbols **combo**, **swap** and **mult** as in the textbook. Write the 3×3 system which results from each of the following operations.

13. `combo(1,3,-1)`

14. `combo(2,3,-5)`

15. `combo(3,2,4)`

16. `combo(2,1,4)`

17. `combo(1,2,-1)`

18. combo(1,2,-e^2)

19. `mult(1,5)`

20. `mult(1,-3)`

21. `mult(2,5)`

22. `mult(2,-2)`

23. `mult(3,4)`

24. `mult(3,5)`

25. mult(2,-π)

26. mult(2,π)

27. mult(1,e^2)

28. mult(1,-e^{-2})

29. `swap(1,3)`

30. `swap(1,2)`

31. `swap(2,3)`

32. `swap(2,1)`

33. `swap(3,2)`

34. `swap(3,1)`

Unique Solution. Create a frame sequence for each system, whose final frame displays the unique solution of the system of equations.

35. $\left|\begin{array}{l} x_1+3x_2=0 \\ x_2=-1 \end{array}\right|$

36. $\left|\begin{array}{l} x_1+2x_2=0 \\ x_2=-2 \end{array}\right|$

37. $\left|\begin{array}{l} x_1+3x_2=2 \\ x_1-x_2=1 \end{array}\right|$

38. $\left|\begin{array}{l} x_1+x_2=-1 \\ x_1+2x_2=-2 \end{array}\right|$

39. $\left|\begin{array}{l} x_1+3x_2+2x_3=1 \\ x_2+4x_3=3 \\ 4x_3=4 \end{array}\right|$

40. $\left|\begin{array}{l} x_1=1 \\ 3x_1+x_2=0 \\ 2x_1+2x_2+3x_3=3 \end{array}\right|$

41. $\left|\begin{array}{l} x_1+x_2+3x_3=1 \\ x_2=2 \\ 3x_3=0 \end{array}\right|$

42. $\left|\begin{array}{l} x_1+3x_2+2x_3=1 \\ x_2=3 \\ 3x_3=0 \end{array}\right|$

43. $\left|\begin{array}{l} x_1=2 \\ x_1+2x_2=1 \\ 2x_1+2x_2+x_3=0 \\ 3x_1+6x_2+x_3+2x_4=2 \end{array}\right|$

44. $\left|\begin{array}{l} x_1=3 \\ x_1-2x_2=1 \\ 2x_1+2x_2+x_3=0 \\ 3x_1+6x_2+x_3+4x_4=2 \end{array}\right|$

45. $\left|\begin{array}{l} x_1+x_2=2 \\ x_1+2x_2=1 \\ 2x_1+2x_2+x_3=0 \\ 3x_1+6x_2+x_3+2x_4=2 \end{array}\right|$

46. $\left|\begin{array}{l} x_1-2x_2=3 \\ x_1-x_2=1 \\ 2x_1+2x_2+x_3=0 \\ 3x_1+6x_2+x_3+4x_4=1 \end{array}\right|$

47. $\left|\begin{array}{l} x_1=3 \\ x_1-x_2=1 \\ 2x_1+2x_2+x_3=0 \\ 3x_1+6x_2+x_3+4x_4=1 \\ 3x_1 \quad x_3 \quad +2x_5=1 \end{array}\right|$

48. $\left|\begin{array}{l} x_1=2 \\ x_1-x_2=0 \\ 2x_1+2x_2+x_3=1 \\ 3x_1+6x_2+x_3+3x_4=1 \\ 3x_1+x_3+3x_5=1 \end{array}\right|$

49. $\left|\begin{array}{l} x_1+x_2=2 \\ x_1-x_2=0 \\ 2x_1+2x_2+2x_3=1 \\ 3x_1+6x_2+x_3+2x_4=1 \\ 3x_1+x_3+3x_5=2 \end{array}\right|$

50. $\left|\begin{array}{l} x_1-x_2=3 \\ x_1-2x_2=0 \\ 2x_1+2x_2+x_3=1 \\ 3x_1+6x_2+x_3+3x_4=1 \\ 3x_1+x_3+x_5=3 \end{array}\right|$

No Solution. Develop a frame sequence for each system, whose final frame contains a signal equation (e.g., $0 = 1$), thereby showing that the system has no solution.

51. $\left|\begin{array}{l} x_1+3x_2=0 \\ x_1+3x_2=1 \end{array}\right|$

52. $\left|\begin{array}{l} x_1+2x_2=1 \\ 2x_1+4x_2=2 \end{array}\right|$

53. $\left|\begin{array}{l} x_1+3x_2+2x_3=1 \\ x_2+4x_3=3 \\ x_2+4x_3=4 \end{array}\right|$

54. $\left|\begin{array}{l} x_1=0 \\ 3x_1+x_2+3x_3=1 \\ 2x_1+2x_2+6x_3=0 \end{array}\right|$

55. $\left|\begin{array}{l} x_1+x_2+3x_3=1 \\ x_2=2 \\ x_1+2x_2+3x_3=2 \end{array}\right|$

56. $\left|\begin{array}{l} x_1+3x_2+2x_3=1 \\ x_2+2x_3=3 \\ x_1+5x_3=5 \end{array}\right|$

57. $\left|\begin{array}{l} x_1=2 \\ x_1+2x_2=2 \\ x_1+2x_2+x_3+2x_4=0 \\ x_1+6x_2+x_3+2x_4=2 \end{array}\right|$

58. $\left|\begin{array}{l} x_1=3 \\ x_1-2x_2=1 \\ 2x_1+2x_2+x_3+4x_4=0 \\ 3x_1+6x_2+x_3+4x_4=2 \end{array}\right|$

59. $\left|\begin{array}{l} x_1 = 3 \\ x_1 - x_2 = 1 \\ 2x_1 + 2x_2 + x_3 = 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 - x_5 = 1 \\ - 6x_2 - x_3 - 4x_4 + x_5 = 0 \end{array}\right|$

60. $\left|\begin{array}{l} x_1 = 3 \\ x_1 - x_2 = 1 \\ 3x_1 + 2x_2 + x_3 = 0 \\ 3x_1 + 6x_2 + x_3 + 4x_4 - x_5 = 1 \\ - 6x_2 - x_3 - 4x_4 + x_5 = 2 \end{array}\right|$

Infinitely Many Solutions.

Display a frame sequence for each system, whose final frame has this property: *each nonzero equation has a lead variable.* Then apply the **last frame algorithm** to write out the standard general solution of the system. Assume in each system variable list x_1 to x_5.

61. $\left|\begin{array}{l} x_1 + x_2 + 3x_3 = 0 \\ x_2 + x_4 = 0 \\ 0 = 0 \end{array}\right|$

62. $\left|\begin{array}{l} x_1 + x_3 = 0 \\ x_1 + x_2 + x_3 + 3x_5 = 0 \\ x_4 + 2x_5 = 0 \end{array}\right|$

63. $\left|\begin{array}{l} x_2 + 3x_3 = 0 \\ x_4 = 0 \\ 0 = 0 \end{array}\right|$

64. $\left|\begin{array}{l} x_1 + 2x_2 + 3x_3 = 0 \\ x_4 = 0 \\ 0 = 0 \end{array}\right|$

65. $\left|\begin{array}{l} x_1 + 2x_2 + 3x_3 = 0 \\ x_3 + x_4 \ 0 = 0 \end{array}\right|$

66. $\left|\begin{array}{l} x_1 + x_2 = 0 \\ x_2 + x_3 = 0 \\ x_3 \ 0 = 1 \end{array}\right|$

67. $\left|\begin{array}{l} x_1 + x_2 + 3x_3 + 5x_4 + 2x_5 = 0 \\ x_5 = 0 \end{array}\right|$

68. $\left|\begin{array}{l} x_1 + 2x_2 + x_3 + 3x_4 + 4x_5 = 0 \\ x_3 + x_4 + x_5 = 0 \end{array}\right|$

69. $\left|\begin{array}{l} x_3 + 2x_4 + x_5 = 0 \\ 2x_3 + 2x_4 + 2x_5 = 0 \\ x_5 = 0 \end{array}\right|$

70. $\left|\begin{array}{l} x_4 + x_5 = 0 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{array}\right|$

71. $\left|\begin{array}{l} x_2 + x_3 + 5x_4 = 0 \\ x_3 + 2x_4 = 0 \\ x_5 = 0 \\ 0 = 0 \end{array}\right|$

72. $\left|\begin{array}{l} x_1 + 3x_3 = 0 \\ x_1 + x_2 + x_4 = 0 \\ x_5 = 0 \\ 0 = 0 \end{array}\right|$

Inverses of Elementary Operations.

Given the final frame of a sequence is

$$\left|\begin{array}{rcrcrcr} 3x & + & 2y & + & 4z & = & 2 \\ x & + & 3y & + & 2z & = & -1 \\ 2x & + & y & + & 5z & = & 0 \end{array}\right|$$

and the given operations, find the original system in the first frame.

73. `combo(1,2,-1)`, `combo(2,3,-3)`, `mult(1,-2)`, `swap(2,3)`.

74. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,2)`, `swap(3,2)`.

75. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`.

76. `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`.

77. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`, `swap(2,3)`.

78. `swap(2,3)`, `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`.

79. `combo(1,2,-1)`, `combo(2,3,3)`, `mult(1,4)`, `swap(1,3)`, `mult(2,3)`.

80. `combo(1,2,-1)`, `combo(2,3,4)`, `mult(1,3)`, `swap(3,2)`, `combo(2,3,-3)`.

3.3 General Solution Theory

Consider the nonhomogeneous system

$$(12)\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

The **general solution** of system (12) is an expression which represents all possible solutions of the system.

The example above for infinitely many solutions contained an unmotivated algorithm which expressed the general solution in terms of invented symbols t_1, t_2, ..., which in mathematical literature are called **parameters**. We outline here some topics from calculus which form the assumed background for this subject.

Equations for Points, Lines and Planes

Background from analytic geometry appears in Table 6. In this table, t_1 and t_2 are **parameters**, which means they are allowed to take on any value between $-\infty$ and $+\infty$. The algebraic equations describing the geometric objects are called **parametric equations**.

Table 6. Parametric equations with geometrical significance.

$x = d_1$, $y = d_2$, $z = d_3$.	**Point**. The equations have no parameters and describe a single point.
$x = d_1 + a_1t_1$, $y = d_2 + a_2t_1$, $z = d_3 + a_3t_1$.	**Line**. The equations with parameter t_1 describe a straight line through (d_1, d_2, d_3) with tangent vector $a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}$.
$x = d_1 + a_1t_1 + b_1t_2$, $y = d_2 + a_2t_1 + b_2t_2$, $z = d_3 + a_3t_1 + b_3t_2$.	**Plane**. The equations with parameters t_1, t_2 describe a plane containing (d_1, d_2, d_3). The cross product $(a_1\vec{\imath} + a_2\vec{\jmath} + a_3\vec{k}) \times (b_1\vec{\imath} + b_2\vec{\jmath} + b_3\vec{k})$ is normal to the plane.

To illustrate, the parametric equations $x = 2 - 6t_1$, $y = -1 - t_1$, $z = 8t_1$ describe the unique line of intersection of the three planes

$$(13)\qquad \begin{aligned} x + 2y + z &= 0, \\ 2x - 4y + z &= 8, \\ 3x - 2y + 2z &= 8. \end{aligned}$$

Details appear in Example 5.

General Solutions

Definition 2 (Parametric Equations)
Equations of the form

$$(14)\qquad \begin{aligned} x_1 &= d_1 + c_{11}t_1 + \cdots + c_{1k}t_k, \\ x_2 &= d_2 + c_{21}t_1 + \cdots + c_{2k}t_k, \\ &\ \vdots \\ x_n &= d_n + c_{n1}t_1 + \cdots + c_{nk}t_k \end{aligned}$$

are called **parametric equations** for the variables $x_1, \ldots, x_n$.

The numbers $d_1, \ldots, d_n, c_{11}, \ldots, c_{nk}$ are *known constants* and the symbols $t_1, \ldots, t_k$ are **parameters**, which are treated as variables that may be assigned any value from $-\infty$ to ∞.

Three cases appear often in examples and exercises, illustrated here for variables x_1, x_2, x_3:

No parameters	One parameter	Two parameters
$x_1 = d_1$	$x_1 = d_1 + a_1t_1$	$x_1 = d_1 + a_1t_1 + b_1t_2$
$x_2 = d_2$	$x_2 = d_2 + a_2t_1$	$x_2 = d_2 + a_2t_1 + b_2t_2$
$x_3 = d_3$	$x_3 = d_3 + a_3t_1$	$x_3 = d_3 + a_3t_1 + b_3t_2$

Definition 3 (General Solution)
A **general solution** of a linear algebraic system of equations (12) is a set of parametric equations (14) plus two additional requirements:

(15) Equations (14) satisfy (5) for all real values of $t_1, \ldots, t_k$.

(16) Any solution of (12) can be obtained from (14) by specializing values of the parameters $t_1, t_2, \ldots t_k$.

A general solution is sometimes called a **parametric solution**. Requirement (15) means that **the solution works**. Requirement (16) means that **no solution was skipped**.

Definition 4 (Standard General Solution)
Parametric equations (14) are called **standard** if they satisfy for distinct subscripts $j_1, i_2, \ldots, j_k$ the equations

$$(17)\qquad x_{j_1} = t_1, \quad x_{j_2} = t_2, \quad \ldots, \quad x_{j_k} = t_k.$$

The relations mean that the full set of parameter symbols $t_1, t_2, \ldots, t_k$ were assigned to k distinct variable names (the **free variables**) selected from $x_1, \ldots, x_n$.

A **standard general solution** of system (12) is a special set of parametric equations (14) satisfying (15), (16) and additionally (17). Toolkit sequences always produce a standard general solution.

Theorem 2 (Standard General Solution)
A standard general solution has the fewest possible parameters and it represents each solution of the linear system by a unique set of parameter values.

The theorem supplies the theoretical basis for the method of toolkit sequences, which formally appears as an algorithm on page 197. The proof of Theorem 2 is delayed until page 220. It is unusual if this proof is a subject of a class lecture, due to its length; it is recommended reading for the mathematically inclined, after understanding the examples.

Reduced Echelon System

Consider a sequence of toolkit operations and the corresponding toolkit sequence. The last frame, from which we write the general solution, is called a reduced echelon system.

Definition 5 (Reduced Echelon System)
A linear system in which each nonzero equation has a **lead variable** is called a **reduced echelon system**. Implicitly assumed are the following definitions and rules.

- A **lead variable** is a variable which appears with coefficient one in the very first location, left to right, in *exactly one* equation.
- A variable not used as a lead variable is called a **free variable**. Variables that do not appear at all are free variables.
- The nonzero equations are listed in variable list order, inherited from their lead variables. Equations without variables are listed last.
- All variables in an equation are required to appear in variable list order. Therefore, within an equation, all free variables are to the right of the lead variable.

Detecting a Reduced Echelon System. A given system can be rapidly inspected to detect if it can be transformed into a reduced echelon system. We assume that within each equation, variables appear in variable list order.

> A nonhomogeneous linear system is recognized as a reduced echelon system when the first variable listed in each equation has coefficient one and that symbol appears nowhere else in the system of equations.[2]

[2] Children are better at such classifications than adults. A favorite puzzle among kids is a drawing which contains disguised figures, like a bird, a fire hydrant and Godzilla. Routinely, children find all the disguised figures.

Such a system can be re-written, by swapping equations and enforcing the rules above, so that the resulting system is a reduced echelon system.

Rank and Nullity

A reduced echelon system splits the variable names $x_1, \ldots, x_n$ into the **lead variables** and the **free variables**. Because the entire variable list is exhausted by these two sets, then

$$\textbf{lead variables} + \textbf{free variables} = \textbf{total variables}.$$

Definition 6 (Rank and Nullity)
The **number of lead variables** in a reduced echelon system is called the **rank** of the system. The number of free variables in a reduced echelon system is called the **nullity** of the system.

Determining rank and nullity. First, display a toolkit sequence which starts with that system and ends in a reduced echelon system. Then the rank and nullity of the system are those determined by the final frame.

Theorem 3 (Rank and Nullity)
The following equation holds:

$$\textbf{rank} + \textbf{nullity} = \text{number of variables.}$$

Computers and Reduced Echelon Form

Computer algebra systems and computer numerical laboratories compute from a given linear system (5) a new equivalent system of identical size, which is called the **reduced row-echelon form**, abbreviated **rref**.

The computed **rref** will pass the *last frame test*, provided there is no signal equation, hence the **rref** is generally a reduced echelon system. This fact is the basis of answer checks with computer assist.

Computer assist requires **matrix input** of the data, a topic which is delayed until a later chapter. Popular commercial programs used to perform the computer assist are `maple`, `mathematica` and `matlab`.

Elimination

The elimination algorithm applies at each algebraic step one of the three toolkit rules defined in Table 1: **swap**, **multiply** and **combination**.

The objective of each algebraic step is to **increase the number of lead variables**. Equivalently, each algebraic step tries to **eliminate**

one repetition of a variable name, which justifies calling the process the **method of elimination**. The process of elimination stops when a signal equation (typically $0 = 1$) is found. Otherwise, elimination stops when no more lead variables can be found, and then the last system of equations is a **reduced echelon system**. A detailed explanation of the process has been given above in the discussion of toolkit sequences.

Reversibility of the algebraic steps means that no solutions are created nor destroyed during the algebra: the original system and all intermediate systems have *exactly the same solutions*.

The final reduced echelon system has either a unique solution or infinitely many solutions, in both cases we report the **general solution**. In the infinitely many solution case, the **last frame algorithm** on page 189 is used to write out a general solution.

Theorem 4 (Elimination)
Every linear system (5) has either no solution or else it has exactly the same solutions as an equivalent reduced echelon system, obtained by repeated use of toolkit rules **swap**, **multiply** and **combination** (page 177).

An Elimination Algorithm

An equation is said to be **processed** if it has a lead variable. Otherwise, the equation is said to be **unprocessed**.

The acronym `rref` abbreviates the phrase *reduced row echelon form*. This abbreviation appears in matrix literature, so we use it instead of creating an acronym for *reduced echelon form* (the word *row* is missing).

1. If an equation "$0 = 0$" appears, then move it to the end. If a signal equation "$0 = c$" appears ($c \neq 0$ required), then the system is inconsistent. In this case, the algorithm halts and we report **no solution**.
2. Identify the **first symbol** x_r, in variable list order $x_1, \ldots, x_n$, which appears in some unprocessed equation. Apply the **multiply** rule to insure x_r has leading coefficient one. Apply the **combination** rule to eliminate variable x_r from all other equations. Then x_r is a **lead variable**: the number of lead variables has been increased by one.
3. Apply the **swap** rule repeatedly to move this equation past all processed equations, but before the unprocessed equations. Mark the equation as **processed**, e.g., replace x_r by boxed symbol $\boxed{x_r}$.
4. Repeat steps 1–3, until all equations have been processed once. Then lead variables $x_{i_1}, \ldots, x_{i_m}$ have been defined and the last system is a reduced echelon system.

Uniqueness, Lead Variables and RREF

Elimination performed on a given system by two different persons will result in the same reduced echelon system. The answer is unique, because attention has been paid to the natural order $x_1, \ldots, x_n$ of the variable list. Uniqueness results from *critical step* **2**, also called the **rref step**:

> Always select a lead variable as the next possible variable name in the original list order $x_1, \ldots, x_n$, taken from all possible unprocessed equations.

This step insures that the final system is a **reduced echelon system**.

The wording **next possible** must be used, because once a variable name is used for a lead variable it may not be used again. The next variable following the last–used lead variable, from the list $x_1, \ldots, x_n$, might not appear in any unprocessed equation, in which case it is a **free variable**. The next variable name in the original list order is then tried as a lead variable.

Numerical Optimization

It is common for references to divide the effort for obtaining an **rref** into two stages, for which the second stage is **back-substitution**. This division of effort is motivated by numerical efficiency considerations, largely historical. The reader is advised to adopt the numerical point of view in hand calculations, as soon as possible. It changes the details of a toolkit sequence to the **rref**: most readers find the changes equally advantageous. The reason for the algorithm in the text is motivational: to become an expert, you have to first *know what you are trying to accomplish*. Exactly how to implement the toolkit to arrive at the **rref** will vary for each person. The recommendation can be phrased as follows:

> Don't bother to eliminate a lead variable from equations already assigned a lead variable. Go on to select the next lead variable and remove that variable from subsequent equations. Final elimination of lead variables from previous equations is saved for the end, then done in reverse variable list order (called **back-substitution**).

Avoiding Fractions

Integer arithmetic should be used, when possible, to speed up hand computation in elimination. To avoid fractions, the **rref** step **2** may be modified to read *with leading coefficient nonzero*. The final division to obtain leading coefficient one is then delayed until the last possible moment.

Examples and Methods

5 Example (Line of Intersection) Show that the parametric equations $x = 2 - 6t$, $y = -1 - t$, $z = 8t$ represent a line through $(2, -1, 0)$ with tangent $-6\vec{\imath} - \vec{\jmath}$ which is the line of intersection of the three planes

$$\begin{array}{rcrcrcl} x & + & 2y & + & z & = & 0, \\ 2x & - & 4y & + & z & = & 8, \\ 3x & - & 2y & + & 2z & = & 8. \end{array} \tag{18}$$

Solution: Using $t = 0$ in the parametric solution shows that $(2, -1, 0)$ is on the line. The tangent to the parametric curve is $x'(t)\vec{\imath} + y'(t)\vec{\jmath} + z'(t)\vec{k}$, which computes to $-6\vec{\imath} - \vec{\jmath}$. The details for showing the parametric solution satisfies the three equations simultaneously:

$$\begin{aligned} \text{LHS} &= x + 2y + z && \text{First equation left side.} \\ &= (2 - 6t) + 2(-1 - t) + 8t && \text{Substitute parametric solution.} \\ &= 0 && \text{Matches the RHS in (18).} \\ \text{LHS} &= 2x - 4y + z && \text{Second equation left side.} \\ &= 2(2 - 6t) - 4(-1 - t) + 8t && \text{Substitute.} \\ &= 8 && \text{Matches (18).} \\ \text{LHS} &= 3x - 2y + 2z && \text{Third equation left side.} \\ &= 3(2 - 6t) - 2(-1 - t) + 16t && \text{Substitute.} \\ &= 8 && \text{Matches (18).} \end{aligned}$$

6 Example (Geometry of Solutions) Solve the system and interpret the solution geometrically.

$$\begin{array}{rcrcl} x & + & 2z & = & 3, \\ y & + & z & = & 1. \end{array}$$

Solution: We begin by displaying the general solution, which is a **line**:

$$\begin{array}{rcll} x & = & 3 - 2t_1, & \\ y & = & 1 - t_1, & \\ z & = & t_1, & \quad -\infty < t_1 < \infty. \end{array}$$

In standard xyz-coordinates, this line passes through $(3, 1, 0)$ with tangent direction $-2\vec{\imath} - \vec{\jmath} + \vec{k}$.

Details. To justify this solution, we observe that the first frame equals the last frame, which is a reduced echelon system in variable list order x, y, z. The standard general solution will be obtained from the last frame algorithm.

$$\boxed{\begin{array}{rcrcl} x & + & 2z & = & 3, \\ y & + & z & = & 1. \end{array}}$$

Frame 1 equals the last frame, a reduced echelon system. The lead variables are x, y and the free variable is z.

$$\boxed{\begin{array}{rclcr} x & = & 3 & - & 2z, \\ y & = & 1 & - & z, \\ z & = & t_1. & & \end{array}}$$

Assign to z invented symbol t_1. Solve for lead variables x and y in terms of the free variable z.

$$\boxed{\begin{array}{lllll} x & = & 3 & - & 2t_1, \\ y & = & 1 & - & t_1, \\ z & = & t_1. & & \end{array}}$$

Back-substitute for free variable z. This is the standard general solution. It is geometrically a line, by Table 6.

7 Example (Symbolic Answer Check) Perform an answer check on

$$\begin{array}{lllll} x & + & 2z & = & 3, \\ y & + & z & = & 1, \end{array}$$

for the general solution

$$\begin{array}{lll} x & = & 3 - 2t_1, \\ y & = & 1 - t_1, \\ z & = & t_1, \qquad -\infty < t_1 < \infty. \end{array}$$

Solution: The displayed answer can be checked manually by substituting the symbolic general solution into the equations $x + 2z = 3$, $y + z = 1$, as follows:

$$\begin{array}{lll} x + 2z & = & (3 - 2t_1) + 2(t_1) \\ & = & 3, \\ y + z & = & (1 - t_1) + (t_1) \\ & = & 1. \end{array}$$

Therefore, the two equations are satisfied for all values of the symbol t_1.

Errors and Skipped Solutions. An algebraic error could lead to a claimed solution $x = 3$, $y = 1$, $z = 0$, which also passes the answer check. While it is *true* that $x = 3$, $y = 1$, $z = 0$ is a solution, it is not the general solution. Infinitely many solutions were skipped in the answer check.

General Solution and Free Variables. The number of lead variables is called the **rank**. The number of free variables is called the **nullity**. The basic relation is **rank** + **nullity** = **number of variables**. Computer algebra systems can compute the rank independently, as a double-check against hand computation. This check is useful for discovering skipped solution errors. The **rank** is unaffected by the ordering of variables.

8 Example (Elimination) Solve the system.

$$\begin{array}{lllllllll} w & + & 2x & - & y & + & z & = & 1, \\ w & + & 3x & - & y & + & 2z & = & 0, \\ & & x & & & + & z & = & -1. \end{array}$$

Solution: The answer using the natural variable list order w, x, y, z is the standard general solution

$$\begin{array}{lll} w & = & 3 + t_1 + t_2, \\ x & = & -1 - t_2, \\ y & = & t_1, \\ z & = & t_2, \qquad -\infty < t_1, t_2 < \infty. \end{array}$$

Details. Elimination will be applied to obtain a toolkit sequence whose last frame justifies the reported solution. The details amount to applying the three rules **swap**, **multiply** and **combination** for equivalent equations on page 177 to obtain a last frame which is a reduced echelon system. The standard general solution from the last frame algorithm matches the one reported above.

Let's mark processed equations with a box enclosing the lead variable (w is marked $\boxed{w}$).

$$\begin{array}{rcrcrcrcr} w & + & 2x & - & y & + & z & = & 1 \\ w & + & 3x & - & y & + & 2z & = & 0 \\ & & x & & & + & z & = & -1 \end{array} \qquad \boxed{\mathbf{1}}$$

$$\begin{array}{rcrcrcrcr} w & + & 2x & - & y & + & z & = & 1 \\ 0 & + & x & + & 0 & + & z & = & -1 \\ & & x & & & + & z & = & -1 \end{array} \qquad \boxed{\mathbf{2}}$$

$$\begin{array}{rcrcrcrcr} \boxed{w} & + & 2x & - & y & + & z & = & 1 \\ & & x & & & + & z & = & -1 \\ & & & & & & 0 & = & 0 \end{array} \qquad \boxed{\mathbf{3}}$$

$$\begin{array}{rcrcrcrcr} \boxed{w} & + & 0 & - & y & - & z & = & 3 \\ & & \boxed{x} & & & + & z & = & -1 \\ & & & & & & 0 & = & 0 \end{array} \qquad \boxed{\mathbf{4}}$$

$\boxed{\mathbf{1}}$ Original system. Identify the variable order as w, x, y, z.

$\boxed{\mathbf{2}}$ Choose w as a lead variable. Eliminate w from equation 2 by using `combo(1,2,-1)`.

$\boxed{\mathbf{3}}$ The w-equation is processed. Let x be the next lead variable. Eliminate x from equation 3 using `combo(2,3,-1)`.

$\boxed{\mathbf{4}}$ Eliminate x from equation 1 using `combo(2,1,-2)`. Mark the x-equation as processed. **Reduced echelon system** found.

The four frames make the **toolkit sequence** which takes the original system into a reduced echelon system. Basic exposition rules apply:

1. Variables in an equation appear in variable list order.
2. Equations inherit variable list order from the lead variables.

The last frame of the sequence, which must be a reduced echelon system, is used to write out the general solution, using the last frame algorithm.

$$\begin{array}{rcrcrcr} \boxed{w} & = & 3 & + & y & + & z \\ \boxed{x} & = & -1 & - & z & & \\ y & = & t_1 & & & & \\ z & = & t_2 & & & & \end{array}$$

Solve for the lead variables $\boxed{w}$, $\boxed{x}$. Assign invented symbols t_1, t_2 to the free variables y, z.

$$\boxed{\begin{array}{rcrcrcr} w &=& 3 &+& t_1 &+& t_2 \\ x &=& -1 &-& t_2 && \\ y &=& t_1 &&&& \\ z &=& t_2 &&&& \end{array}}$$

Back-substitute free variables into the lead variable equations to get a standard general solution.

Answer check. The check will be performed according to the outline on page 218. The justification for this forward reference is to illustrate how to check answers without using the invented symbols t_1, t_2, ... in the details.

Step 1. The **nonhomogeneous trial solution** $w = 3$, $x = -1$, $y = z = 0$ is obtained by setting $t_1 = t_2 = 0$. It is required to satisfy the nonhomogeneous system

$$\begin{array}{rcrcrcrcr} w &+& 2x &-& y &+& z &=& 1, \\ w &+& 3x &-& y &+& 2z &=& 0, \\ && x &&&+& z &=& -1. \end{array}$$

Step 2. The partial derivatives ∂_{t_1}, ∂_{t_2} are applied to the parametric solution to obtain two homogeneous trial solutions $w = 1$, $x = 0$, $y = 1$, $z = 0$ and $w = 1$, $x = -1$, $y = 0$, $z = 1$, which are required to satisfy the homogeneous system

$$\begin{array}{rcrcrcrcr} w &+& 2x &-& y &+& z &=& 0, \\ w &+& 3x &-& y &+& 2z &=& 0, \\ && x &&&+& z &=& 0. \end{array}$$

Each trial solution from **Step 1** and **Step 2** is checked by direct substitution. The method uses **superposition** in order to eliminate the invented symbols from the answer check.

9 Example (No solution) Verify by applying elimination that the system has no solution.

$$\begin{array}{rcrcrcrcr} w &+& 2x &-& y &+& z &=& 0, \\ w &+& 3x &-& y &+& 2z &=& 0, \\ && x &&&+& z &=& 1. \end{array}$$

Solution: Elimination (page 198) will be applied, using the toolkit rules **swap**, **multiply** and **combination** (page 177).

$$\boxed{\begin{array}{rcrcrcrcr} w &+& 2x &-& y &+& z &=& 0 \\ w &+& 3x &-& y &+& 2z &=& 0 \\ && x &&&+& z &=& 1 \end{array}}$$

1

$$\boxed{\begin{array}{rcrcrcrcr} \boxed{w} &+& 2x &-& y &+& z &=& 0 \\ 0 &+& x &+& 0 &+& z &=& 0 \\ && x &&&+& z &=& 1 \end{array}}$$

2

$$\boxed{\begin{array}{rcrcrcrcr} \boxed{w} &+& 2x &-& y &+& z &=& 0 \\ && x &&&+& z &=& 0 \\ &&&&&& 0 &=& 1 \end{array}}$$

3

1 Original system. Select variable order w, x, y, z. Identify lead variable w.

2 Eliminate w from other equations using `combo(1,2,-1)`. Mark the w-equation processed with $\boxed{w}$.

3 Identify lead variable x. Then eliminate x from the third equation using `combo(2,3,-1)`. **Signal equation** found.

The appearance of the signal equation "$0 = 1$" means **no solution**. The logic: if the original system has a solution, then so does the present equivalent system, hence $0 = 1$, a contradiction. Elimination halts, because of the **inconsistent system** containing the false equation "$0 = 1$."

10 Example (Reduced Echelon form) Find an equivalent system in reduced echelon form.

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & x_3 & + & x_4 & = & 1, \\ x_1 & + & 3x_2 & - & x_3 & + & 2x_4 & = & 0, \\ & & x_2 & & & + & x_4 & = & -1. \end{array}$$

Solution: The answer using the natural variable list order x_1, x_2, x_2, x_4 is the non-homogeneous system in **reduced echelon form** (briefly, **rref** form)

$$\begin{array}{rcrcrcrcr} x_1 & & & - & x_3 & - & x_4 & = & 3 \\ & & x_2 & & & + & x_4 & = & -1 \\ & & & & & & 0 & = & 0 \end{array}$$

The **lead variables** are x_1, x_2 and the **free variables** are x_3, x_4. The standard general solution of this system is

$$\begin{array}{rcl} x_1 & = & 3 + t_1 + t_2, \\ x_2 & = & -1 - t_2, \\ x_3 & = & t_1, \\ x_4 & = & t_2, \qquad\qquad -\infty < t_1, t_2 < \infty. \end{array}$$

The details are the same as Example 8, with $w = x_1$, $x = x_2$, $y = x_3$, $z = x_4$. The toolkit sequence has three frames and the last frame is used to display the general solution.

Answer check in `maple`. The output from the `maple` code below duplicates the reduced echelon system reported above and the general solution.

```
with(LinearAlgebra):
 eq1:=x[1]+2*x[2]-x[3]+x[4]=1:eq2:=x[1]+3*x[2]-x[3]+2*x[4]=0:
 eq3:=x[2]+x[4]=-1:eqs:=[eq1,eq2,eq3]:var:=[x[1],x[2],x[3],x[4]]:
 A:=GenerateMatrix(eqs,var,augmented);
 F:=ReducedRowEchelonForm(A);
 GenerateEquations(F,var);
 F,LinearSolve(F,free=t); # general solution answer check
 A,LinearSolve(A,free=t); # general solution answer check
```

Exercises 3.3

Classification. Classify the parametric equations as a point, line or plane, then compute as appropriate the tangent to the line or the normal to the plane.

1. $x = 0,\ y = 1,\ z = -2$

2. $x = 1,\ y = -1,\ z = 2$

3. $x = t_1,\ y = 1 + t_1,\ z = 0$

4. $x = 0,\ y = 0,\ z = 1 + t_1$

5. $x = 1 + t_1,\ y = 0,\ z = t_2$

6. $x = t_2 + t_1,\ y = t_2,\ z = t_1$

7. $x = 1,\ y = 1 + t_1,\ z = 1 + t_2$

8. $x = t_2 + t_1,\ y = t_1 - t_2,\ z = 0$

9. $x = t_2,\ y = 1 + t_1,\ z = t_1 + t_2$

10. $x = 3t_2 + t_1,\ y = t_1 - t_2,\ z = 2t_1$

Reduced Echelon System. Solve the xyz–system and interpret the solution geometrically.

11. $\left|\begin{array}{rcl} y + z &=& 1 \\ x + 2z &=& 2 \end{array}\right|$

12. $\left|\begin{array}{rcl} x + z &=& 1 \\ y + 2z &=& 4 \end{array}\right|$

13. $\left|\begin{array}{rcl} y + z &=& 1 \\ x + 3z &=& 2 \end{array}\right|$

14. $\left|\begin{array}{rcl} x + z &=& 1 \\ y + z &=& 5 \end{array}\right|$

15. $\left|\begin{array}{rcl} x + z &=& 1 \\ 2x + 2z &=& 2 \end{array}\right|$

16. $\left|\begin{array}{rcl} x + y &=& 1 \\ 3x + 3y &=& 3 \end{array}\right|$

17. $\left|\ x + y + z = 1.\ \right|$

18. $\left|\ x + 2y + 4z = 0.\ \right|$

19. $\left|\begin{array}{rcl} x + y &=& 2 \\ z &=& 1 \end{array}\right|$

20. $\left|\begin{array}{rcl} x + 4z &=& 0 \\ y &=& 1 \end{array}\right|$

Homogeneous System. Solve the xyz–system using elimination with variable list order x, y, z.

21. $\left|\begin{array}{rcl} y + z &=& 0 \\ 2x + 2z &=& 0 \end{array}\right|$

22. $\left|\begin{array}{rcl} x + z &=& 0 \\ 2y + 2z &=& 0 \end{array}\right|$

23. $\left|\begin{array}{rcl} x + z &=& 0 \\ 2z &=& 0 \end{array}\right|$

24. $\left|\begin{array}{rcl} y + z &=& 0 \\ y + 3z &=& 0 \end{array}\right|$

25. $\left|\begin{array}{rcl} x + 2y + 3z &=& 0 \\ 0 &=& 0 \end{array}\right|$

26. $\left|\begin{array}{rcl} x + 2y &=& 0 \\ 0 &=& 0 \end{array}\right|$

27. $\left|\begin{array}{rcl} y + z &=& 0 \\ 2x + 2z &=& 0 \\ x + z &=& 0 \end{array}\right|$

28. $\left|\begin{array}{rcl} 2x + y + z &=& 0 \\ x + 2z &=& 0 \\ x + y - z &=& 0 \end{array}\right|$

29. $\left|\begin{array}{rcl} x + y + z &=& 0 \\ 2x + 2z &=& 0 \\ x + z &=& 0 \end{array}\right|$

30. $\left|\begin{array}{rcl} x + y + z &=& 0 \\ 2x + 2z &=& 0 \\ 3x + y + 3z &=& 0 \end{array}\right|$

Nonhomogeneous 3×3 System. Solve the xyz-system using elimination and variable list order x, y, z.

31. $\left|\begin{array}{rcl} y &=& 1 \\ 2z &=& 2 \end{array}\right|$

32. $\left|\begin{array}{rcl} x &=& 1 \\ 2z &=& 2 \end{array}\right|$

33. $$\left|\begin{array}{rcl} y + z &=& 1 \\ 2x + 2z &=& 2 \\ x + z &=& 1 \end{array}\right|$$

34. $$\left|\begin{array}{rcl} 2x + y + z &=& 1 \\ x + 2z &=& 2 \\ x + y - z &=& -1 \end{array}\right|$$

35. $$\left|\begin{array}{rcl} x + y + z &=& 1 \\ 2x + 2z &=& 2 \\ x + z &=& 1 \end{array}\right|$$

36. $$\left|\begin{array}{rcl} x + y + z &=& 1 \\ 2x + 2z &=& 2 \\ 3x + y + 3z &=& 3 \end{array}\right|$$

37. $$\left|\begin{array}{rcl} 2x + y + z &=& 3 \\ 2x + 2z &=& 2 \\ 4x + y + 3z &=& 5 \end{array}\right|$$

38. $$\left|\begin{array}{rcl} 2x + y + z &=& 2 \\ 6x \quad y + 5z &=& 2 \\ 4x + y + 3z &=& 2 \end{array}\right|$$

39. $$\left|\begin{array}{rcl} 6x + 2y + 6z &=& 10 \\ 6x \quad y + 6z &=& 11 \\ 4x + y + 4z &=& 7 \end{array}\right|$$

40. $$\left|\begin{array}{rcl} 6x + 2y + 4z &=& 6 \\ 6x \quad y + 5z &=& 9 \\ 4x + y + 3z &=& 5 \end{array}\right|$$

Nonhomogeneous 3×4 System. Solve the $yzuv$-system using elimination with variable list order y, z, u, v.

41. $$\left|\begin{array}{rcl} y + z + 4u + 8v &=& 10 \\ 2z - u + v &=& 10 \\ 2y - u + 5v &=& 10 \end{array}\right|$$

42. $$\left|\begin{array}{rcl} y + z + 4u + 8v &=& 10 \\ 2z - 2u + 2v &=& 0 \\ y + 3z + 2u + 5v &=& 5 \end{array}\right|$$

43. $$\left|\begin{array}{rcl} y + z + 4u + 8v &=& 1 \\ 2z - 2u + 4v &=& 0 \\ y + 3z + 2u + 6v &=& 1 \end{array}\right|$$

44. $$\left|\begin{array}{rcl} y + 3z + 4u + 8v &=& 1 \\ 2z - 2u + 4v &=& 0 \\ y + 3z + 2u + 6v &=& 1 \end{array}\right|$$

45. $$\left|\begin{array}{rcl} y + 3z + 4u + 8v &=& 1 \\ 2z - 2u + 4v &=& 0 \\ y + 4z + 2u + 7v &=& 1 \end{array}\right|$$

46. $$\left|\begin{array}{rcl} y + z + 4u + 9v &=& 1 \\ 2z - 2u + 4v &=& 0 \\ y + 4z + 2u + 7v &=& 1 \end{array}\right|$$

47. $$\left|\begin{array}{rcl} y + z + 4u + 9v &=& 1 \\ 2z - 2u + 4v &=& 0 \\ y + 4z + 2u + 7v &=& 1 \end{array}\right|$$

48. $$\left|\begin{array}{rcl} y + z + 4u + 9v &=& 10 \\ 2z - 2u + 4v &=& 4 \\ y + 4z + 2u + 7v &=& 8 \end{array}\right|$$

49. $$\left|\begin{array}{rcl} y + z + 4u + 9v &=& 2 \\ 2z - 2u + 4v &=& 4 \\ y + 3z + 5u + 13v &=& 0 \end{array}\right|$$

50. $$\left|\begin{array}{rcl} y + z + 4u + 3v &=& 2 \\ 2z - 2u + 4v &=& 4 \\ y + 3z + 5u + 7v &=& 0 \end{array}\right|$$

3.4 Basis, Dimension, Nullity and Rank

Studied here are the basic concepts of rank, nullity, basis and dimension for a system of linear algebraic equations.

Definition 7 (Rank and Nullity)
The **rank** of a system of linear algebraic equations is the number of lead variables appearing in its reduced echelon form. The **nullity** of a system of linear algebraic equations is the number of free variables.

> **rank** = number of lead variables
> **nullity** = number of free variables
> **rank + nullity = number of variables**

Definition 8 (Basis and Dimension)
Consider a homogeneous system of linear algebraic equations. A list of k solutions of the system is called a **basis** provided

1. The general solution of the system can be constructed from the list of k solutions.

2. The list size k cannot be decreased.

The **dimension** of the system of linear algebraic equations is the unique number k satisfying **1** and **2**. The dimension equals the minimum number of invented symbols used in any general solution, which also equals the nullity.

> A **basis** is an alternate representation of the general solution which has **no invented symbols**.

Basis Illustration

Consider the homogeneous system

$$\begin{aligned} x + 2y + 3z &= 0, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}$$

It is a reduced echelon system with standard general solution

$$\begin{aligned} x &= -2t_1 - 3t_2, \\ y &= t_1, \\ z &= t_2. \end{aligned}$$

The formal partial derivatives ∂_{t_1}, ∂_{t_2} of the general solution are solutions of the homogeneous system, because they correspond exactly to setting $t_1 = 1$, $t_2 = 0$ and $t_1 = 0$, $t_2 = 1$, respectively:

$$\begin{array}{llll} x = -2, & y = 1, & z = 0, & \text{(partial on } t_1) \\ x = -3, & y = 0, & z = 1. & \text{(partial on } t_2) \end{array}$$

A **basis** for the homogeneous system is the list of two solutions displayed above. A general solution of the homogeneous system can be re-constructed from this basis by multiplying the first solution by invented symbol t_1 and the second solution by invented symbol t_2, then add to obtain

$$\begin{aligned} x &= -2t_1 - 3t_2, \\ y &= t_1, \\ z &= t_2. \end{aligned}$$

This display is the original standard general solution, reconstructed from the list of solutions in the basis.

Non-uniqueness of a Basis. A given homogeneous linear system has a number of different standard general solutions, obtained, for example, by re-ordering the variable list. Therefore, a *basis is not unique.* Language like *the basis* is tragically incorrect.

To illustrate non-uniqueness, consider the homogeneous 3×3 system of equations

$$\begin{array}{rcl} x + y + z & = & 0, \\ 0 & = & 0, \\ 0 & = & 0. \end{array} \tag{19}$$

Equations (19) have two standard general solutions

$x = -t_1 - t_2$, $y = t_1$, $z = t_2$
and
$x = t_3$, $y = -t_3 - t_4$, $z = t_4$,

corresponding to two different orderings of the variable list x, y, z. Then **two different bases** for the system are given by the partial derivative relations

$$\partial_{t_1}, \partial_{t_2} : \quad \left\{ \begin{array}{llll} x = -1, & y = 1, & z = 0, & \text{Basis 1,} \\ x = -1, & y = 0, & z = 1, & \end{array} \right. \tag{20}$$

$$(21) \qquad \partial_{t_3}, \partial_{t_4} : \quad \begin{cases} x = 1, & y = -1, & z = 0, & \text{Basis 2,} \\ x = 0, & y = -1, & z = 1. \end{cases}$$

In general, there are *infinitely many bases* possible for a given linear homogeneous system.

Nullspace

Definition 9 (Nullspace)
Consider a system of linear homogeneous algebraic equations. The term **nullspace** refers to the set of all solutions to the system. A synonym for nullspace is the term **kernel**.

Why null? The prefix **null** refers to the right side of the homogeneous system, which is zero, or *null*, for each equation. The main reason for introducing the term **nullspace** is to consider simultaneously *all possible* general solutions of the linear system, without regard to their representation in terms of invented symbols or the algorithm used to find the formulas.

No Geometry. The term **nullspace** uses the word **space**, which has meaning taken from the phrases **storage space** and **parking space** — it has no intended geometrical meaning whatsoever.

How to Find the Nullspace. A classical method for describing the nullspace is to form a toolkit sequence for the homogeneous system which ends with a reduced echelon system. The last frame algorithm applies to write the general solution in terms of invented symbols t_1, t_2, ... The meaning is that assignment of values to the symbols t_1, t_2, ... lists all possible solutions of the system. The general solution formula obtained by this method is one possible set of scalar equations that completely describes all solutions of the homogeneous equation, hence it describes completely the nullspace.

Basis for the Nullspace. A **basis** for the nullspace is found by taking partial derivatives ∂_{t_1}, ∂_{t_2}, ... on invented symbols $t_1, t_2, \ldots$ in the last frame algorithm general solution, giving k solutions.[3] The general solution is reconstructed from these basis elements by multiplying them by the symbols t_1, t_2, ... and adding. The nullspace is the same regardless of the choice of basis, because it is just the set of solutions of the homogeneous equation.

[3] Gilbert Strang calls the answers *special solutions*. We will call them **Strang's special solutions**, for lack of a better name.

An Illustration. Consider the system

$$
(22)\qquad \begin{aligned} x + y + 2z &= 0, \\ 0 &= 0, \\ 0 &= 0. \end{aligned}
$$

The nullspace is the set of all solutions of $x + y + 2z = 0$. Geometrically, it is the plane $x + y + 2z = 0$ through $x = y = z = 0$ with normal vector $\vec{\imath} + \vec{\jmath} + 2\vec{k}$. The nullspace is represented by the general solution formula

$$
\begin{aligned} x &= -t_1 - 2t_2, \\ y &= t_1, \\ z &= t_2. \end{aligned}
$$

There are infinitely many representations possible, e.g., replace t_1 by mt_1 where m is any nonzero integer.

The nullspace can be described succinctly as the plane generated by the basis

$$
\begin{aligned} x = -1, y = 1, z = 0, \\ x = -2, y = 0, z = 1. \end{aligned}
$$

Calculus courses represent the two basis elements as vectors $\vec{\mathbf{a}} = -\vec{\imath} + \vec{\jmath}$, $\vec{\mathbf{b}} = -2\vec{\imath} + \vec{k}$, which are two vectors in the plane $x + y + 2z = 0$. Their cross product $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ is normal to the plane, a multiple of $\vec{\imath} + \vec{\jmath} + 2\vec{k}$.

The Three Possibilities Revisited

We intend to justify the table below, which summarizes the three possibilities for a linear system, in terms of free variables, rank and nullity.

Table 7. Three possibilities for an $m \times n$ linear system.

No solution	Signal equation	
∞-many solutions	One+ free variables	nullity ≥ 1 or rank $< n$
Unique solution	Zero free variables	nullity $= 0$ or rank $= n$

No Solution. There is no solution to a system of equations exactly when a signal equation $0 = 1$ occurs during the application of swap, multiply and combination rules. We report the system **inconsistent** and announce **no solution**.

Infinitely Many Solutions. The situation of infinitely many solutions occurs when there is no signal equation and **at least one free variable** to which an invented symbol, say t_1, is assigned. Since this symbol takes the values $-\infty < t_1 < \infty$, there are an infinity of solutions. The conditions **rank less than n** and **nullity positive** are the same.

Unique Solution. There is a unique solution to a consistent system of equations exactly when **zero free variables** are present. This is identical to requiring that the number n of variables equal the number of lead variables, or **rank = n**.

Existence of Infinitely Many Solutions

Homogeneous systems are always consistent[4], therefore if the number of variables exceeds the number of equations, then the equation **lead** + **free** = **variable count** implies there is always one free variable. This proves the following basic result of linear algebra.

Theorem 5 (Infinitely Many Solutions)
A system of $m \times n$ linear homogeneous equations (6) with fewer equations than unknowns ($m < n$) has at least one free variable, hence an infinite number of solutions. Therefore, such a system always has the **zero solution** and also a **nonzero solution**.

Non-homogeneous systems can be similarly analyzed by considering conditions under which there will be at least one free variable.

Theorem 6 (Missing Variable and Infinitely Many Solutions)
A consistent system of $m \times n$ linear equations with one unknown missing has at least one free variable, hence an infinite number of solutions.

Theorem 7 (Rank, Nullity and Infinitely Many Solutions)
A consistent system of $m\times n$ linear equations with nonzero nullity or rank less than n has at least one free variable, hence an infinite number of solutions.

Examples and Methods

11 **Example (Rank and Nullity)** Determine using an abbreviated sequence of toolkit operations the rank and nullity of the homogeneous system

$$\begin{array}{rrrl} x_1 & + 4x_3 & + 8x_4 & = 0 \\ & - x_3 & + x_4 & = 0 \\ 2x_1 & - x_3 & + 5x_4 & = 0 \end{array}$$

Solution: The answer is three (3) lead variables and one (1) free variable, making rank=3 and nullity=1.

The missing variable x_2 implies that there is at least one free variable. The abbreviated steps are

$$\boxed{\begin{array}{rrrl} x_1 & + 4x_3 & + 8x_4 & = 0 \\ & - x_3 & + x_4 & = 0 \\ & - 9x_3 & - 11x_4 & = 0 \end{array}} \qquad \texttt{combo(1,3,-2)}$$

[4] All variables set to zero is always a solution of a homogeneous system.

$$\boxed{\begin{array}{rrrl} x_1 & +\ 4x_3 & +\ \ 8x_4 & = 0 \\ & -\ \ x_3 & +\ \ \ x_4 & = 0 \\ & & -\ 20x_4 & = 0 \end{array}}$$ `combo(2,3,-9)`

The triangular form implies that x_1, x_3, x_4 are lead variables and x_2 is a free variable.

12 Example (Nullspace Basis or Kernel Basis) Determine a nullspace basis by solving for the general solution of the homogeneous system

$$\begin{array}{rrrrl} x_1 + & x_2 & + 4x_3 & + 9x_4 & = 0 \\ & 2x_2 & - x_3 & + 4x_4 & = 0 \end{array}$$

Solution:

$$\boxed{\begin{array}{rrrrl} x_1 + & x_2 & + 4x_3 & + 9x_4 & = 0 \\ & 2x_2 & - x_3 & + 4x_4 & = 0 \end{array}}$$ Original system.

$$\boxed{\begin{array}{rrrrl} x_1 + & x_2 & + 4x_3 & + 9x_4 & = 0 \\ & x_2 & - \frac{1}{2}x_3 & + 2x_4 & = 0 \end{array}}$$ `mult(2,1/2)`

$$\boxed{\begin{array}{rrrrl} x_1 & & + \frac{9}{2}x_3 & + 7x_4 & = 0 \\ & x_2 & - \frac{1}{2}x_3 & + 2x_4 & = 0 \end{array}}$$ `combo(2,1,-1)`

The lead variables are x_1, x_2 and the free variables are $x_3 = t_1, x_4 = t_2$ in terms of invented symbols t_1, t_2. Back-substitution implies the scalar general solution

$$\begin{array}{rcl} x_1 & = & -\frac{9}{2}t_1 - 7t_2, \\ x_2 & = & \frac{1}{2}t_1 - 2t_2, \\ x_3 & = & t_1, \\ x_4 & = & t_2. \end{array} \tag{23}$$

A suitable basis for the **nullspace**, also called the **kernel**, is found by substitution of $t_1 = 1, t_2 = 0$ and then $t_1 = 0, t_2 = 1$, to obtain the two vectors

Basis solution 1	Basis solution 2
$x_1 = -\frac{9}{2}$,	$x_1 = -7$,
$x_2 = \frac{1}{2}$,	$x_2 = -2$,
$x_3 = 1$,	$x_3 = 0$,
$x_4 = 0$.	$x_4 = 1$.

These two solutions are identical to the two solutions obtained by taking partial derivatives ∂_{t_1} and ∂_{t_2} on the scalar general solution displayed in equation (23).

Some references suggest to make the two basis answers fraction-free by choosing t_1, t_2 appropriately. In the present case, this amounts to multiplying the answers by 2. The result is a different basis.

Either answer is sufficient, because a basis is not unique: the only requirement is re-construction of the general solution from the basis.

13 Example (Three Possibilities with Symbol k) Determine all values of the symbol k such that the system below has one of the **Three Possibilities** (1) *No solution*, (2) *Infinitely many solutions* or (3) *A unique solution*. Display all solutions found.

$$\begin{array}{rcrcl} x & + & ky & = & 2, \\ (2-k)x & + & y & = & 3. \end{array}$$

Solution: The Three Possibilities are detected by (1) A signal equation "$0 = 1$," (2) One or more free variables, (3) Zero free variables.

The solution of this problem involves construction of perhaps three toolkit sequences, the last frame of each resulting in one of the three possibilities (1), (2), (3).

$$\begin{array}{rcrcr} x & + & ky & = & 2, \\ (2-k)x & + & y & = & 3. \end{array}$$

Frame 1.

Original system.

$$\begin{array}{rcrcr} x & + & ky & = & 2, \\ & & [1+k(k-2)]y & = & 2(k-2)+3. \end{array}$$

Frame 2.

`combo(1,2,k-2)`

$$\begin{array}{rcrcr} x & + & ky & = & 2, \\ & & (k-1)^2 y & = & 2k-1. \end{array}$$

Frame 3.

Simplify.

The three expected toolkit sequences share these initial frames. At this point, we identify the values of k that split off into the three possibilities.

There will be a signal equation if the second equation of Frame 3 has no variables, but the resulting equation is not "$0 = 0$." This happens exactly for $k = 1$. The resulting signal equation is "$0 = 1$." We conclude that one of the three toolkit sequences terminates with the *no solution case*. This toolkit sequence corresponds to $k = 1$.

Otherwise, $k \neq 1$. For these values of k, there are zero free variables, which implies a unique solution. A by-product of the analysis is that the *infinitely many solutions* case never occurs!

The conclusion: The initially expected three toolkit sequences reduce to two toolkit sequences. One sequence gives no solution and the other sequence gives a unique solution.

The three answers:

(1) No solution occurs only for $k = 1$.

(2) Infinitely many solutions occurs for no value of k.

(3) A unique solution occurs for $k \neq 1$.

$$x = 2 - \frac{k(2k-1)}{(k-1)^2},$$
$$y = \frac{(2k-1)}{(k-1)^2}.$$

14 Example (Symbols and the Three Possibilities) Determine all values of the symbols a, b such that the system below has (1) No solution, (2) Infinitely many solutions or (3) A unique solution. Display all solutions found.

$$\begin{array}{rcrcrcl} x & + & ay & + & bz & = & 2, \\ & & y & + & z & = & 3, \\ & & by & + & z & = & 3b. \end{array}$$

Solution: The plan is to make three toolkit sequences, using swap, multiply and combination rules. Each sequence has last frame which is one of the three possibilities, the detection facilitated by (1) A signal equation "$0 = 1$," (2) At least one free variable, (3) Zero free variables. The initial three frames of each of the expected toolkit sequences is constructed as follows.

System	Comment
$\begin{array}{rcrcrcr} x & + & ay & + & bz & = & 2, \\ & & y & + & z & = & 3, \\ & & by & + & z & = & 3b. \end{array}$	Frame 1 Original system.
$\begin{array}{rcrcrcr} x & + & ay & + & bz & = & 2, \\ & & y & + & z & = & 3, \\ & & 0 & + & (1-b)z & = & 0. \end{array}$	Frame 2. `combo(2,3,-b)`
$\begin{array}{rcrcrcr} x & + & 0 & + & (b-a)z & = & 2-3a, \\ & & y & + & z & = & 3, \\ & & 0 & + & (1-b)z & = & 0. \end{array}$	Frame 3. `combo(2,1,-a)` Triangular form. Lead variables determined.

The three toolkit sequences expected will share these initial frames. Frame 3 shows that there are either 2 lead variables or 3 lead variables, accordingly as the coefficient of z in the third equation is nonzero or zero. There will never be a signal equation. Consequently, the three expected toolkit sequences reduce to just two. We complete these two sequences to give the answer:

(1) There are no values of a, b that result in no solution.

(2) If $1 - b = 0$, then there are two lead variables and hence an infinite number of solutions, given by the general solution

$$\begin{cases} x = 2 - 3a - (b - a)t_1, \\ y = 3 - t_1, \\ z = t_1. \end{cases}$$

(3) If $1 - b \neq 0$, then there are three lead variables and there is a unique solution, given by

$$\begin{cases} x = 2 - 3a, \\ y = 3, \\ z = 0. \end{cases}$$

Exercises 3.4

Rank and Nullity. Compute an abbreviated sequence of `combo`, `swap`, `mult` steps which finds the value of the rank and nullity.

1. $$\left|\begin{array}{l} x_1 + x_2 + 4x_3 + 8x_4 = 0 \\ \quad\; 2x_2 - x_3 + x_4 = 0 \end{array}\right|$$

2. $$\left|\begin{array}{l} x_1 + x_2 + 8x_4 = 0 \\ \quad\; 2x_2 + x_4 = 0 \end{array}\right|$$

3. $$\left|\begin{array}{l} x_1 + 2x_2 + 4x_3 + 9x_4 = 0 \\ x_1 + 8x_2 + 2x_3 + 7x_4 = 0 \end{array}\right|$$

4. $$\left|\begin{array}{l} x_1 + x_2 + 4x_3 + 11x_4 = 0 \\ \quad\; 2x_2 - 2x_3 + 4x_4 = 0 \end{array}\right|$$

Nullspace. Solve using variable order y, z, u, v. Report the values of the **nullity** and **rank** in the equation **nullity+rank**=4.

5. $$\left|\begin{array}{l} y + z + 4u + 8v = 0 \\ \quad\; 2z - u + v = 0 \\ 2y - u + 5v = 0 \end{array}\right|$$

6. $$\left|\begin{array}{l} y + z + 4u + 8v = 0 \\ \quad\; 2z - 2u + 2v = 0 \\ y + 3z + 2u + 5v = 0 \end{array}\right|$$

7. $$\left|\begin{array}{l} y + z + 4u + 8v = 0 \\ \quad\; 2z - 2u + 4v = 0 \\ y + 3z + 2u + 6v = 0 \end{array}\right|$$

8. $$\left|\begin{array}{l} y + 3z + 4u + 8v = 0 \\ \quad\; 2z - 2u + 4v = 0 \\ y + 3z + 2u + 6v = 0 \end{array}\right|$$

9. $$\left|\begin{array}{l} y + 3z + 4u + 8v = 0 \\ \quad\; 2z - 2u + 4v = 0 \end{array}\right|$$

10. $$\left|\begin{array}{l} y + z + 4u + 9v = 0 \\ \quad\; 2z - 2u + 4v = 0 \end{array}\right|$$

11. $$\left|\begin{array}{l} y + z + 4u + 9v = 0 \\ 3y + 4z + 2u + 5v = 0 \end{array}\right|$$

12. $$\left|\begin{array}{l} y + 2z + 4u + 9v = 0 \\ y + 8z + 2u + 7v = 0 \end{array}\right|$$

13. $$\left|\begin{array}{l} y + z + 4u + 11v = 0 \\ \quad\; 2z - 2u + 4v = 0 \end{array}\right|$$

14. $$\left|\begin{array}{l} y + z + 5u + 11v = 0 \\ \quad\; 2z - 2u + 6v = 0 \end{array}\right|$$

Dimension of the nullspace. In the homogeneous systems, assume variable order x, y, z, u, v.

(a) Display an equivalent set of equations in reduced echelon form.

(b) Solve for the general solution and check the answer.

(c) Report the dimension of the nullspace.

15. $$\left|\begin{array}{l} x + y + z + 4u + 8v = 0 \\ -x + 2z - 2u + 2v = 0 \\ \quad\; y - z + 6u + 6v = 0 \end{array}\right|$$

16. $$\left|\begin{array}{l} x + y + z + 4u + 8v = 0 \\ \quad\; - 2z - u + v = 0 \\ \quad\; 2y - u + 5v = 0 \end{array}\right|$$

17. $$\left|\begin{array}{l} \quad\; y + z + 4u + 8v = 0 \\ x + 2z - 2u + 4v = 0 \\ 2x + y + 3z + 2u + 6v = 0 \end{array}\right|$$

18. $$\left|\begin{array}{l} x + y + 3z + 4u + 8v = 0 \\ 2x + 2z - 2u + 4v = 0 \\ x - y + 3z + 2u + 12v = 0 \end{array}\right|$$

19. $$\left|\begin{array}{l} \quad\; y + 3z + 4u + 20v = 0 \\ \quad\; + 2z - 2u + 10v = 0 \\ - y + 3z + 2u + 30v = 0 \end{array}\right|$$

20. $$\left|\begin{array}{l} \quad\; y + 4u + 20v = 0 \\ \quad\; - 2u + 10v = 0 \\ - y + 2u + 30v = 0 \end{array}\right|$$

21. $$\left|\begin{array}{l} x + y + z + 4u = 0 \\ \quad\; - 2z - u = 0 \\ \quad\; 2y - u + = 0 \end{array}\right|$$

22. $$\left|\begin{array}{l} \quad\; + z + 12u + 8v = 0 \\ x + 2z - 6u + 4v = 0 \\ 2x + 3z + 6u + 6v = 0 \end{array}\right|$$

23. $$\left|\begin{array}{l} y + z + 4u = 0 \\ \quad\; 2z - 2u = 0 \\ y - z + 6u = 0 \end{array}\right|$$

24. $\begin{vmatrix} x & + & z & + 8v = 0 \\ & - & 2z & + \; v = 0 \\ & & & 5v = 0 \end{vmatrix}$

Three possibilities with symbols. Assume variables x, y, z. Determine the values of the constants (a, b, c, k, etc) such that the system has (1) *No solution*, (2) *A unique solution* or (3) *Infinitely many solutions*.

25. $\begin{vmatrix} x + ky = 0 \\ x + 2ky = 0 \end{vmatrix}$

26. $\begin{vmatrix} kx + ky = 0 \\ x + 2ky = 0 \end{vmatrix}$

27. $\begin{vmatrix} ax + by = 0 \\ x + 2by = 0 \end{vmatrix}$

28. $\begin{vmatrix} bx + ay = 0 \\ x + 2y = 0 \end{vmatrix}$

29. $\begin{vmatrix} bx + ay = c \\ x + 2y = b - c \end{vmatrix}$

30. $\begin{vmatrix} bx + ay = 2c \\ x + 2y = c + a \end{vmatrix}$

31. $\begin{vmatrix} bx + ay + z = 0 \\ 2bx + ay + 2z = 0 \\ x + 2y + 2z = c \end{vmatrix}$

32. $\begin{vmatrix} bx + ay + z = 0 \\ 3bx + 2ay + 2z = 2c, \\ x + 2y + 2z = c \end{vmatrix}$

33. $\begin{vmatrix} 3x + ay + z = b \\ 2bx + ay + 2z = 0 \\ x + 2y + 2z = c \end{vmatrix}$

34. $\begin{vmatrix} x + ay + z = 2b \\ 3bx + 2ay + 2z = 2c \\ x + 2y + 2z = c \end{vmatrix}$

Three Possibilities. The following questions can be answered by using the quantitative expression of the three possibilities in terms of lead and free variables, rank and nullity.

35. Does there exist a homogeneous 3×2 system with a unique solution? Either give an example or else prove that no such system exists.

36. Does there exist a homogeneous 2×3 system with a unique solution? Either give an example or else prove that no such system exists.

37. In a homogeneous 10×10 system, two equations are identical. Prove that the system has a nonzero solution.

38. In a homogeneous 5×5 system, each equation has a leading variable. Prove that the system has only the zero solution.

39. Suppose given two homogeneous systems A and B, with A having a unique solution and B having infinitely many solutions. Explain why B cannot be obtained from A by a sequence of swap, multiply and combination operations on the equations.

40. A 2×3 system cannot have a unique solution. Cite a theorem or explain why.

41. If a 3×3 homogeneous system contains no variables, then what is the general solution?

42. If a 3×3 non-homogeneous solution has a unique solution, then what is the nullity of the homogeneous system?

43. A 7×7 homogeneous system is missing two variables. What is the maximum rank of the system? Give examples for all possible ranks.

44. Suppose an $n \times n$ system of equations (homogeneous or non-homogeneous) has two solutions. Prove that it has infinitely many solutions.

45. What is the nullity and rank of an $n \times n$ system of homogeneous equations if the system has a unique solution?

46. What is the nullity and rank of an $n \times n$ system of non-homogeneous equations if the system has a unique solution?

47. Prove or disprove (by example): A 4×3 nonhomogeneous system cannot have a unique solution.

48. Prove or disprove (by example): A 4×3 homogeneous system always has infinitely many solutions.

3.5 Answer Check, Proofs and Details

Answer Check Algorithm

A given general solution (14) can be tested for validity manually as in Example 6, page 200. It is possible to devise a **symbol-free answer check**. The technique checks a general solution (14) by testing constant trial solutions in systems (5) and (6).

Step 1. Set all invented symbols $t_1, \ldots, t_k$ to zero in general solution (14) to obtain the nonhomogeneous trial solution $x_1 = d_1$, $x_2 = d_2, \ldots, x_n = d_n$. Test it by direct substitution into the nonhomogeneous system (5).

Step 2. Apply partial derivatives $\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_k}$ to the general solution (14), obtaining k homogeneous trial solutions. Verify that the trial solutions satisfy the homogeneous system (6), by direct substitution.

The trial solutions in **step 2** are obtained from the general solution (14) by setting one invented symbol equal to 1 and the others zero, followed by subtracting the nonhomogeneous trial solution of **step 1**. The partial derivative idea computes the same set of trial solutions, and it is easier to remember.

Theorem 8 (Answer Check)
The answer check algorithm described in steps 1–2 verifies a solution (14) for all values of the symbols. Please observe that this answer check cannot test for skipped solutions.

Proof of Theorem 8. To simplify notation and quickly communicate the ideas, a proof will be given for a 2×2 system. A proof for the $m \times n$ case can be constructed by the reader, using the same ideas. Consider the nonhomogeneous and homogeneous systems

$$\begin{aligned} ax_1 + by_1 &= b_1, \\ cx_1 + dy_1 &= b_2, \end{aligned} \tag{24}$$

$$\begin{aligned} ax_2 + by_2 &= 0, \\ cx_2 + dy_2 &= 0. \end{aligned} \tag{25}$$

Assume (x_1, y_1) is a solution of (24) and (x_2, y_2) is a solution of (25). Add corresponding equations in (24) and (25). Then collecting terms gives

$$\begin{aligned} a(x_1 + x_2) + b(y_1 + y_2) &= b_1, \\ c(x_1 + x_2) + d(y_1 + y_2) &= b_2. \end{aligned} \tag{26}$$

This proves that $(x_1 + x_2, y_1 + y_2)$ is a solution of the nonhomogeneous system. Similarly, a scalar multiple (kx_2, ky_2) of a solution (x_2, y_2) of system (25) is also a solution of (25) and the sum of two solutions of (25) is again a solution of (25).

Given each solution in **step 2** satisfies (25), then multiplying the first solution by t_1 and the second solution by t_2 and adding gives a solution (x_3, y_3) of (25). After adding (x_3, y_3) to the solution (x_1, y_1) of **step 1**, a solution of (24) is obtained, proving that the full parametric solution containing the symbols t_1, t_2 is a solution of (24). The proof for the 2×2 case is complete.

Failure of Answer Checks

An answer check only tests the given formulas against the equations. If too few parameters are present, then the answer check can be algebraically correct but the general solution check fails, because not all solutions can be obtained by specialization of the parameter values.

For example, $x = 1 - t_1$, $y = t_1$, $z = 0$ is a one-parameter solution for $x + y + z = 1$, as verified by an answer check. But the general solution $x = 1 - t_1 - t_2$, $y = t_1$, $z = t_2$ has two parameters t_1, t_2. Generally, an answer check decides if the formula supplied works in the equation. It does **not** decide if the given formula represents **all** solutions. This trouble, in which an error leads to a *smaller* value for the nullity of the system, is due largely to human error and not machine error.

Linear algebra workbenches have another kind of flaw: they may compute the **nullity** for a system incorrectly as an integer *larger* than the correct nullity. A parametric solution with nullity k might be obtained, checked to work in the original equations, then cross-checked by computing the nullity k independently. However, the computed nullity k could be greater than the actual nullity of the system. Here is a simple example, where ϵ is a *very* small positive number:

$$\begin{array}{rcl} x + y &=& 0, \\ \epsilon y &=& \epsilon. \end{array} \qquad (27)$$

On a limited precision machine, system (27) has internal machine representation[5]

$$\begin{array}{rcl} x + y &=& 0, \\ 0 &=& 0. \end{array} \qquad (28)$$

Representation (28) occurs because the coefficient ϵ is smaller than the smallest positive floating point number of the machine, hence it becomes zero during translation. System (27) has nullity zero and system (28) has nullity one. The parametric solution for system (28) is $x = -t_1$, $y = t_1$, with basis selected by setting $t_1 = 1$. The basis passes the answer check on system (27), because ϵ times 1 evaluates to ϵ. A second check

[5]For example, if the machine allows only 2-digit exponents (10^{99} is the maximum), then $\epsilon = 10^{-101}$ translates to zero.

for the nullity of system (28) gives 1, which supports the correctness of the parametric solution, but unfortunately there are not infinitely many solutions: for system (27) the correct answer is the unique solution $x = -1$, $y = 1$.

Computer algebra systems (CAS) are supposed to avoid this kind of error, because they do not translate input into floating point representations. All input is supposed to remain in symbolic or in string form. In short, they don't change ϵ to zero. Because of this standard, CAS are safer systems in which to do linear algebra computations, albeit slower in execution.

The trouble reported here is not entirely one of input translation. An innocuous `combo(1,2,-1)` can cause an equation like $\epsilon y = \epsilon$ in the middle of a toolkit sequence. If floating point hardware is being used, and not symbolic computation, then the equation can translate to $0 = 0$, causing a false free variable appearance.

Minimal Parametric Solutions

Proof of Theorem 2: The proof of Theorem 2, page 196, will follow from the lemma and theorem below.

Lemma 1 (Unique Representation) If a set of parametric equations (14) satisfies (15), (16) and (17), then each solution of linear system (5) is given by (14) for exactly one set of parameter values.

Proof: Let a solution of system (5) be given by (14) for two sets of parameters $t_1, \ldots, t_k$ and $\bar{t}_1, \ldots, \bar{t}_k$. By (17), $t_j = x_{i_j} = \bar{t}_j$ for $1 \le j \le k$, therefore the parameter values are the same.

Definition 10 (Minimal Parametric Solution)
Given system (5) has a parametric solution $x_1, \ldots, x_n$ satisfying (14), (15), (16), then among all such parametric solutions there is one which uses the *fewest* possible parameters. A parametric solution with fewest parameters is called **minimal**. Parametric solutions with more parameters are called **redundant**.

To illustrate, the plane $x+y+z = 1$ has a minimal standard parametric solution $x = 1-t_1-t_2$, $y = t_1$, $z = t_2$. A redundant parametric solution of $x+y+z = 1$ is $x = 1 - t_1 - t_2 - 2t_3$, $y = t_1 + t_3$, $z = t_2 + t_3$, using three parameters t_1, t_2, t_3.

Theorem 9 (Minimal Parametric Solutions)
Let linear system (5) have a parametric solution satisfying (14), (15), (16). Then (14) has the fewest possible parameters if and only if each solution of linear system (5) is given by (14) for exactly one set of parameter values.

Proof: Suppose first that a general solution (14) is given with the least number k of parameters, but contrary to the theorem, there are two ways to represent some solution, with corresponding parameters $r_1, \ldots, r_k$ and also $s_1, \ldots, s_k$. Subtract the two sets of parametric equations, thus eliminating the symbols x_1, $\ldots$, x_n, to obtain:

$$\begin{aligned} c_{11}(r_1 - s_1) + \cdots + c_{1k}(r_k - s_k) &= 0, \\ &\vdots \\ c_{n1}(r_1 - s_1) + \cdots + c_{nk}(r_k - s_k) &= 0. \end{aligned}$$

Relabel the variables and constants so that $r_1 - s_1 \neq 0$, possible since the two sets of parameters are supposed to be different. Divide the preceding equations by $r_1 - s_1$ and solve for the constants $c_{11}, \ldots, c_{n1}$. This results in equations

$$\begin{aligned} c_{11} &= c_{12}w_2 + \cdots + c_{1k}w_k, \\ &\vdots \\ c_{n1} &= c_{n2}w_2 + \cdots + c_{nk}w_k, \end{aligned}$$

where $w_j = -\frac{r_j - s_j}{r_1 - s_1}$, $2 \leq j \leq k$. Insert these relations into (14), effectively eliminating the symbols $c_{11}, \ldots, c_{n1}$, to obtain

$$\begin{aligned} x_1 &= d_1 + c_{12}(t_2 + w_2t_1) + \cdots + c_{1k}(t_k + w_kt_1), \\ x_2 &= d_2 + c_{22}(t_2 + w_2t_1) + \cdots + c_{2k}(t_k + w_kt_1), \\ &\vdots \\ x_n &= d_n + c_{n2}(t_2 + w_2t_1) + \cdots + c_{nk}(t_k + w_kt_1). \end{aligned}$$

Let $t_1 = 0$. The remaining parameters $t_2, \ldots, t_k$ are fewer parameters that describe all solutions of the system, a contradiction to the definition of k. This completes the proof of the first half of the theorem.

To prove the second half of the theorem, assume that a parametric solution (14) is given which represents all possible solutions of the system and in addition each solution is represented by exactly one set of parameter values. It will be established that the number k in (14) is the *least possible* parameter count.

Suppose not. Then there is a second parametric solution

$$\begin{aligned} x_1 &= e_1 + b_{11}v_1 + \cdots + b_{1\ell}v_\ell, \\ &\vdots \\ x_n &= e_n + b_{n1}v_1 + \cdots + b_{n\ell}v_\ell, \end{aligned} \tag{29}$$

where $\ell < k$ and $v_1, \ldots, v_\ell$ are the parameters. It is assumed that (29) represents all solutions of the linear system.

We shall prove that the solutions for zero parameters in (14) and (29) can be taken to be the same, that is, another parametric solution is given by

$$\begin{aligned} x_1 &= d_1 + b_{11}s_1 + \cdots + b_{1\ell}s_\ell, \\ &\vdots \\ x_n &= d_n + b_{n1}s_1 + \cdots + b_{n\ell}s_\ell. \end{aligned} \tag{30}$$

The idea of the proof is to substitute $x_1 = d_1, \ldots, x_n = d_n$ into (29) for parameters $r_1, \ldots, r_n$. Then solve for $e_1, \ldots, e_n$ and replace back into (29) to obtain

$$\begin{aligned} x_1 &= d_1 + b_{11}(v_1 - r_1) + \cdots + b_{1\ell}(v_\ell - r_\ell), \\ &\vdots \\ x_n &= d_n + b_{n1}(v_1 - r_1) + \cdots + b_{n\ell}(v_\ell - r_\ell). \end{aligned}$$

Replacing parameters $s_j = v_j - r_j$ gives (30).

From (14) it is known that $x_1 = d_1 + c_{11}, \ldots, x_n = d_n + c_{n1}$ is a solution. By (30), there are constants $r_1, \ldots, r_\ell$ such that (we cancel $d_1, \ldots, d_n$ from both sides)

$$\begin{aligned} c_{11} &= b_{11}r_1 + \cdots + b_{1\ell}r_\ell, \\ &\vdots \\ c_{n1} &= b_{n1}r_1 + \cdots + b_{n\ell}r_\ell. \end{aligned}$$

If r_1 through r_ℓ are all zero, then the solution just referenced equals $d_1, \ldots, d_n$, hence (14) has a solution that can be represented with parameters all zero or with $t_1 = 1$ and all other parameters zero, a contradiction. Therefore, some $r_i \neq 0$ and we can assume by renumbering that $r_1 \neq 0$. Return now to the last system of equations and divide by r_1 in order to solve for the constants $b_{11}, \ldots, b_{n1}$. Substitute the answers back into (30) in order to obtain parametric equations

$$\begin{aligned} x_1 &= d_1 + c_{11}w_1 + b_{12}w_2 + \cdots + b_{1\ell}w_\ell, \\ &\vdots \\ x_n &= d_n + c_{n1}w_1 + b_{n2}w_2 + \cdots + b_{n\ell}w_\ell, \end{aligned}$$

where $w_1 = s_1$, $w_j = s_j - r_j/r_1$. Given $s_1, \ldots, s_\ell$ are parameters, then so are $w_1, \ldots, w_\ell$.

This process can be repeated for the solution $x_1 = d_1 + c_{12}, \ldots, x_n = d_n + c_{n2}$. We assert that for some index j, $2 \le j \le \ell$, constants $b_{ij}, \ldots, b_{nj}$ in the previous display can be isolated, and the process of replacing symbols b by c continued. If not, then $w_2 = \cdots = w_\ell = 0$. Then solution $x_1, \ldots, x_n$ has two distinct representations in (14), first with $t_2 = 1$ and all other $t_j = 0$, then with $t_1 = w_1$ and all other $t_j = 0$. A contradiction results, which proves the assertion. After ℓ repetitions of this replacement process, we find a parametric solution

$$\begin{aligned} x_1 &= d_1 + c_{11}u_1 + c_{12}u_2 + \cdots + c_{1\ell}u_\ell, \\ &\vdots \\ x_n &= d_n + c_{n1}u_1 + c_{n2}u_2 + \cdots + c_{n\ell}u_\ell, \end{aligned}$$

in some set of parameters $u_1, \ldots, u_\ell$.

However, $\ell < k$, so at least the solution $x_1 = d_1 + c_{1k}, \ldots, x_n = d_n + c_{nk}$ remains unused by the process. Insert this solution into the previous display, valid for some parameters $u_1, \ldots, u_\ell$. The relation says that the solution $x_1 = d_1, \ldots, x_n = d_n$ in (14) has two distinct sets of parameters, namely $t_1 = u_1, \ldots, t_\ell = u_\ell$, $t_k = -1$, all others zero, and also all parameters zero, a contradiction. This completes the proof of the theorem.

Exercises 3.5

Parametric solutions.

1. Is there a 2×3 homogeneous system with general solution having 2 parameters t_1, t_2?

2. Is there a 3×3 homogeneous system with general solution having 3 parameters t_1, t_2, t_3?

3. Give an example of a 4×3 homogeneous system with general solution having zero parameters, that is, $x = y = z = 0$ is the only solution.

4. Give an example of a 4×3 homogeneous system with general solution having exactly one parameter t_1.

5. Give an example of a 4×3 homogeneous system with general solution having exactly two parameters t_1, t_2.

6. Give an example of a 4×3 homogeneous system with general solution having exactly three parameters t_1, t_2, t_3.

7. Consider an $n \times n$ homogeneous system with parametric solution having parameters t_1 to t_k. What are the possible values of k?

8. Consider an $n \times m$ homogeneous system with parametric solution having parameters t_1 to t_k. What are the possible values of k?

Answer Checks. Assume variable list x, y, z and parameter t_1. (a) Display the answer check details. (b) Find the rank. (c) Report whether the given solution is a general solution.

9. $\left|\begin{array}{rcl} y & = & 1 \\ 2z & = & 2 \end{array}\right|$
$x = t_1, y = 1, z = 1$.

10. $\left|\begin{array}{rcl} x & = & 1 \\ 2z & = & 2 \end{array}\right|$
$x = 1, y = t_1, z = 1$.

11. $\left|\begin{array}{rcl} y + z & = & 1 \\ 2x + 2z & = & 2 \\ x + z & = & 1 \end{array}\right|$
$x = 0, y = 0, z = 1$.

12. $\left|\begin{array}{rcl} 2x + y + z & = & 1 \\ x + 2z & = & 2 \\ x + y - z & = & -1 \end{array}\right|$
$x = 2, y = -3, z = 0$.

13. $\left|\begin{array}{rcl} x + y + z & = & 1 \\ 2x + 2z & = & 2 \\ x + z & = & 1 \end{array}\right|$
$x = 1 - t_1, y = 0, z = t_1$.

14. $\left|\begin{array}{rcl} x + y + z & = & 1 \\ 2x + 2z & = & 2 \\ 3x + y + 3z & = & 3 \end{array}\right|$
$x = 1 - t_1, y = 0, z = t_1$.

Failure of Answer Checks. Find the unique solution for $\epsilon > 0$. Discuss how a machine might translate the system to obtain infinitely many solutions.

15. $x + \epsilon y = 1$, $x - \epsilon y = 1$

16. $x + y = 1$, $x + (1 + \epsilon)y = 1 + \epsilon$

17. $x + \epsilon y = 10\epsilon$, $x - \epsilon y = 10\epsilon$

18. $x + y = 1 + \epsilon$, $x + (1 + \epsilon)y = 1 + 11\epsilon$

Minimal Parametric Solutions. For each given system, determine if the expression is a minimal general solution.

19. $\left|\begin{array}{l} y + z + 4u + 8v = 0 \\ 2z - u + v = 0 \\ 2y - u + 5v = 0 \end{array}\right|$
$y = -3t_1, z = -t_1$,
$u = -t_1, v = t_1$.

20. $\left|\begin{array}{l} y + z + 4u + 8v = 0 \\ 2z - 2u + 2v = 0 \\ y - z + 6u + 6v = 0 \end{array}\right|$
$y = -5t_1 - 7t_2, z = t_1 - t_2$,
$u = t_1, v = t_2$.

21. $\left|\begin{array}{l} y + z + 4u + 8v = 0 \\ 2z - 2u + 4v = 0 \\ y + 3z + 2u + 6v = 0 \end{array}\right|$
$y = -5t_1 + 5t_2, z = t_1 - t_2$,
$u = t_1 - t_2, v = 0$.

22. $\left|\begin{array}{l} y + 3z + 4u + 8v = 0 \\ 2z - 2u + 4v = 0 \\ y + 3z + 2u + 12v = 0 \end{array}\right|$
$y = 5t_1 + 4t_2, z = -3t_1 - 6t_2$,
$u = -t_1 - 2t_2, v = t_1 + 2t_2$.