is of the form in (47) with

$$\mathbf{P}(t) = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} -9t + 13 \\ 7t - 15 \\ -6t + 7 \end{bmatrix}.$$

In Example 7 we saw that a general solution of the associated homogeneous linear system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \mathbf{x}$$

is given by

$$\mathbf{x}_c(t) = \begin{bmatrix} 2c_1e^t + 2c_2e^{3t} + 2c_3e^{5t} \\ 2c_1e^t \qquad\qquad - 2c_3e^{5t} \\ c_1e^t - c_2e^{3t} + c_2e^{5t} \end{bmatrix},$$

and we can verify by substitution that the function

$$\mathbf{x}_p(t) = \begin{bmatrix} 3t \\ 5 \\ 2t \end{bmatrix}$$

(found using a computer algebra system, or perhaps by a human being using a method discussed in Section 5.6) is a particular solution of the original nonhomogeneous system. Consequently, Theorem 4 implies that a general solution of the nonhomogeneous system is given by

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t);$$

that is, by

$$\begin{aligned} x_1(t) &= 2c_1e^t + 2c_2e^{3t} + 2c_3e^{5t} + 3t, \\ x_2(t) &= 2c_1e^t \qquad\qquad - 2c_3e^{5t} + 5, \\ x_3(t) &= c_1e^t - c_2e^{3t} + c_3e^{5t} + 2t. \end{aligned}$$

■

5.1 Problems

1. Let

$$\mathbf{A} = \begin{bmatrix} 2 & -3 \\ 4 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 3 & -4 \\ 5 & 1 \end{bmatrix}.$$

Find **(a)** $2\mathbf{A} + 3\mathbf{B}$; **(b)** $3\mathbf{A} - 2\mathbf{B}$; **(c)** $\mathbf{AB}$; **(d)** $\mathbf{BA}$.

2. Verify that (a) $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ and that (b) $\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC}$, where $\mathbf{A}$ and $\mathbf{B}$ are the matrices given in Problem 1 and

$$\mathbf{C} = \begin{bmatrix} 0 & 2 \\ 3 & -1 \end{bmatrix}.$$

3. Find $\mathbf{AB}$ and $\mathbf{BA}$ given

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ 3 & -4 & 5 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 1 & 3 \\ -7 & 0 \\ 3 & -2 \end{bmatrix}.$$

4. Let $\mathbf{A}$ and $\mathbf{B}$ be the matrices given in Problem 3 and let

$$\mathbf{x} = \begin{bmatrix} 2t \\ e^{-t} \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} t^2 \\ \sin t \\ \cos t \end{bmatrix}.$$

Find $\mathbf{Ay}$ and $\mathbf{Bx}$. Are the products $\mathbf{Ax}$ and $\mathbf{By}$ defined? Explain your answer.

5. Let

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 4 & 3 \\ -5 & 2 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0 & -3 & 2 \\ 1 & 4 & -3 \\ 2 & 5 & -1 \end{bmatrix}.$$

Find **(a)** $7\mathbf{A} + 4\mathbf{B}$; **(b)** $3\mathbf{A} - 5\mathbf{B}$; **(c)** $\mathbf{AB}$; **(d)** $\mathbf{BA}$; **(e)** $\mathbf{A} - t\mathbf{I}$.

6. Let

$$\mathbf{A}_1=\begin{bmatrix}2 & 1\\ -3 & 2\end{bmatrix},\quad \mathbf{A}_2=\begin{bmatrix}1 & 3\\ -1 & -2\end{bmatrix},$$
$$\mathbf{B}=\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}.$$

(a) Show that $\mathbf{A}_1\mathbf{B}=\mathbf{A}_2\mathbf{B}$ and note that $\mathbf{A}_1\neq\mathbf{A}_2$. Thus the cancellation law does not hold for matrices; that is, if $\mathbf{A}_1\mathbf{B}=\mathbf{A}_2\mathbf{B}$ and $\mathbf{B}\neq\mathbf{0}$, it does not follow that $\mathbf{A}_1=\mathbf{A}_2$. (b) Let $\mathbf{A}=\mathbf{A}_1-\mathbf{A}_2$ and use part (a) to show that $\mathbf{AB}=\mathbf{0}$. Thus the product of two nonzero matrices may be the zero matrix.

7. Compute the determinants of the matrices **A** and **B** in Problem 6. Are your results consistent with the theorem to the effect that

$$\det(\mathbf{AB})=\det(\mathbf{A})\cdot\det(\mathbf{B})$$

for any two square matrices **A** and **B** of the same order?

8. Suppose that **A** and **B** are the matrices of Problem 5. Verify that $\det(\mathbf{AB})=\det(\mathbf{BA})$.

In Problems 9 and 10, verify the product law for differentiation, $(\mathbf{AB})'=\mathbf{A}'\mathbf{B}+\mathbf{AB}'$.

9. $\mathbf{A}(t)=\begin{bmatrix}t & 2t-1\\ t^3 & \dfrac{1}{t}\end{bmatrix}$ and $\mathbf{B}(t)=\begin{bmatrix}1-t & 1+t\\ 3t^2 & 4t^3\end{bmatrix}$

10. $\mathbf{A}(t)=\begin{bmatrix}e^t & t & t^2\\ -t & 0 & 2\\ 8t & -1 & t^3\end{bmatrix}$ and $\mathbf{B}(t)=\begin{bmatrix}3\\ 2e^{-t}\\ 3t\end{bmatrix}$

In Problems 11 through 20, write the given system in the form $\mathbf{x}'=\mathbf{P}(t)\mathbf{x}+\mathbf{f}(t)$.

11. $x'=-3y,\ y'=3x$
12. $x'=3x-2y,\ y'=2x+y$
13. $x'=2x+4y+3e^t,\ y'=5x-y-t^2$
14. $x'=tx-e^ty+\cos t,\ y'=e^{-t}x+t^2y-\sin t$
15. $x'=y+z,\ y'=z+x,\ z'=x+y$
16. $x'=2x-3y,\ y'=x+y+2z,\ z'=5y-7z$
17. $x'=3x-4y+z+t,\ y'=x-3z+t^2,\ z'=6y-7z+t^3$
18. $x'=tx-y+e^tz,\ y'=2x+t^2y-z,\ z'=e^{-t}x+3ty+t^3z$
19. $x_1'=x_2,\ x_2'=2x_3,\ x_3'=3x_4,\ x_4'=4x_1$
20. $x_1'=x_2+x_3+1,\ x_2'=x_3+x_4+t,$
$x_3'=x_1+x_4+t^2,\ x_4'=x_1+x_2+t^3$

In Problems 21 through 30, first verify that the given vectors are solutions of the given system. Then use the Wronskian to show that they are linearly independent. Finally, write the general solution of the system.

21. $\mathbf{x}'=\begin{bmatrix}4 & 2\\ -3 & -1\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=\begin{bmatrix}2e^t\\ -3e^t\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}e^{2t}\\ -e^{2t}\end{bmatrix}$

22. $\mathbf{x}'=\begin{bmatrix}-3 & 2\\ -3 & 4\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=\begin{bmatrix}e^{3t}\\ 3e^{3t}\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}2e^{-2t}\\ e^{-2t}\end{bmatrix}$

23. $\mathbf{x}'=\begin{bmatrix}3 & -1\\ 5 & -3\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=e^{2t}\begin{bmatrix}1\\ 1\end{bmatrix},\ \mathbf{x}_2=e^{-2t}\begin{bmatrix}1\\ 5\end{bmatrix}$

24. $\mathbf{x}'=\begin{bmatrix}4 & 1\\ -2 & 1\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=e^{3t}\begin{bmatrix}1\\ -1\end{bmatrix},\ \mathbf{x}_2=e^{2t}\begin{bmatrix}1\\ -2\end{bmatrix}$

25. $\mathbf{x}'=\begin{bmatrix}4 & -3\\ 6 & -7\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=\begin{bmatrix}3e^{2t}\\ 2e^{2t}\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}e^{-5t}\\ 3e^{-5t}\end{bmatrix}$

26. $\mathbf{x}'=\begin{bmatrix}3 & -2 & 0\\ -1 & 3 & -2\\ 0 & -1 & 3\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=e^t\begin{bmatrix}2\\ 2\\ 1\end{bmatrix},$
$\mathbf{x}_2=e^{3t}\begin{bmatrix}-2\\ 0\\ 1\end{bmatrix},\ \mathbf{x}_3=e^{5t}\begin{bmatrix}2\\ -2\\ 1\end{bmatrix}$

27. $\mathbf{x}'=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=e^{2t}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix},$
$\mathbf{x}_2=e^{-t}\begin{bmatrix}1\\ 0\\ -1\end{bmatrix},\ \mathbf{x}_3=e^{-t}\begin{bmatrix}0\\ 1\\ -1\end{bmatrix}$

28. $\mathbf{x}'=\begin{bmatrix}1 & 2 & 1\\ 6 & -1 & 0\\ -1 & -2 & -1\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=\begin{bmatrix}1\\ 6\\ -13\end{bmatrix},$
$\mathbf{x}_2=e^{3t}\begin{bmatrix}2\\ 3\\ -2\end{bmatrix},\ \mathbf{x}_3=e^{-4t}\begin{bmatrix}-1\\ 2\\ 1\end{bmatrix}$

29. $\mathbf{x}'=\begin{bmatrix}-8 & -11 & -2\\ 6 & 9 & 2\\ -6 & -6 & 1\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=e^{-2t}\begin{bmatrix}3\\ -2\\ 2\end{bmatrix},$
$\mathbf{x}_2=e^{t}\begin{bmatrix}1\\ -1\\ 1\end{bmatrix},\ \mathbf{x}_3=e^{3t}\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}$

30. $\mathbf{x}'=\begin{bmatrix}1 & -4 & 0 & -2\\ 0 & 1 & 0 & 0\\ 6 & -12 & -1 & -6\\ 0 & -4 & 0 & -1\end{bmatrix}\mathbf{x};\ \mathbf{x}_1=e^{-t}\begin{bmatrix}1\\ 0\\ 0\\ 1\end{bmatrix},$
$\mathbf{x}_2=e^{-t}\begin{bmatrix}0\\ 0\\ 1\\ 0\end{bmatrix},\ \mathbf{x}_3=e^{t}\begin{bmatrix}0\\ 1\\ 0\\ -2\end{bmatrix},\ \mathbf{x}_4=e^{t}\begin{bmatrix}1\\ 0\\ 3\\ 0\end{bmatrix}$

In Problems 31 through 40, find a particular solution of the indicated linear system that satisfies the given initial conditions.

31. The system of Problem 22: $x_1(0)=0,\ x_2(0)=5$
32. The system of Problem 23: $x_1(0)=5,\ x_2(0)=-3$
33. The system of Problem 24: $x_1(0)=11,\ x_2(0)=-7$
34. The system of Problem 25: $x_1(0)=8,\ x_2(0)=0$
35. The system of Problem 26: $x_1(0)=0,\ x_2(0)=0,$ $x_3(0)=4$
36. The system of Problem 27: $x_1(0)=10,\ x_2(0)=12,$ $x_3(0)=-1$
37. The system of Problem 29: $x_1(0)=1,\ x_2(0)=2,$ $x_3(0)=3$
38. The system of Problem 29: $x_1(0)=5,\ x_2(0)=-7,$ $x_3(0)=11$

39. The system of Problem 30: $x_1(0) = x_2(0) = x_3(0) = x_4(0) = 1$

40. The system of Problem 30: $x_1(0) = 1$, $x_2(0) = 3$, $x_3(0) = 4$, $x_4(0) = 7$

41. **(a)** Show that the vector functions

$$\mathbf{x}_1(t) = \begin{bmatrix} t \\ t^2 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} t^2 \\ t^3 \end{bmatrix}$$

are linearly independent on the real line. **(b)** Why does it follow from Theorem 2 that there is *no* continuous matrix $\mathbf{P}(t)$ such that $\mathbf{x}_1$ and $\mathbf{x}_2$ are both solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$?

42. Suppose that one of the vector functions

$$\mathbf{x}_1(t) = \begin{bmatrix} x_{11}(t) \\ x_{21}(t) \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2(t) = \begin{bmatrix} x_{12}(t) \\ x_{22}(t) \end{bmatrix}$$

is a constant multiple of the other on the open interval I. Show that their Wronskian $W(t) = |[x_{ij}(t)]|$ must vanish identically on I. This proves part (a) of Theorem 2 in the case $n = 2$.

43. Suppose that the vectors $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ of Problem 42 are solutions of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$, where the 2×2 matrix $\mathbf{P}(t)$ is continuous on the open interval I. Show that if there exists a point a of I at which their Wronskian $W(a)$ is zero, then there exist numbers c_1 and c_2 not both zero such that $c_1\mathbf{x}_1(a) + c_2\mathbf{x}_2(a) = \mathbf{0}$. Then conclude from the uniqueness of solutions of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ that

$$c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) = \mathbf{0}$$

for all t in I; that is, that $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly dependent. This proves part (b) of Theorem 2 in the case $n = 2$.

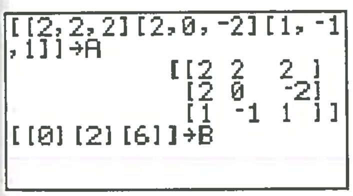
44. Generalize Problems 42 and 43 to prove Theorem 2 for n an arbitrary positive integer.

45. Let $\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_n(t)$ be vector functions whose ith components (for some fixed i) $x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)$ are linearly independent real-valued functions. Conclude that the vector functions are themselves linearly independent.

5.1 Application Automatic Solution of Linear Systems

Linear systems with more than two or three equations are most frequently solved with the aid of calculators or computers. For instance, recall that in Example 8 we needed to solve the linear system

$$\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= 0, \\ 2c_1 \qquad - 2c_3 &= 2, \\ c_1 - c_2 + c_3 &= 6 \end{aligned} \tag{1}$$



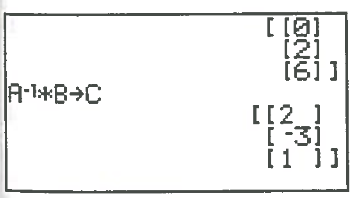


FIGURE 5.1.1. TI-86 solution of the system $\mathbf{AC} = \mathbf{B}$ in (1).

that can be written in the form $\mathbf{AC} = \mathbf{B}$ with 3×3 coefficient matrix $\mathbf{A}$, right-hand side the 3×1 column vector $\mathbf{B} = \begin{bmatrix} 0 & 2 & 6 \end{bmatrix}^T$, and unknown column vector $\mathbf{C} = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^T$. Figure 5.1.1 shows a TI calculator solution for $\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}$, with the result that $c_1 = 2$, $c_2 = -3$, and $c_3 = 1$. The same result can be found using the *Maple* commands

```
with(linalg):
A := array([[2, 2, 2], [2, 0, -2], [1, -1, 1]]):
B := array([[0], [2], [6] ]):
C := multiply(inverse(A),B);
```

the *Mathematica* commands

```
A = {{2, 2, 2}, {2, 0, -2}, {1, -1, 1}};
B = {{0}, {2}, {6}};
C = Inverse[A].B
```

or the MATLAB commands

```
A = [[2  2  2]; [2  0  -2]; [1  -1  1]];
B = [0; 2; 6];
C = inv(A)*B
```

Use your own calculator or available computer algebra system to solve "automatically" Problems 31 through 40 in this section.

5.2 Problems

In Problems 1 through 16, apply the eigenvalue method of this section to find a general solution of the given system. If initial values are given, find also the corresponding particular solution. For each problem, use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

1. $x_1' = x_1 + 2x_2,\ \ x_2' = 2x_1 + x_2$
2. $x_1' = 2x_1 + 3x_2,\ \ x_2' = 2x_1 + x_2$
3. $x_1' = 3x_1 + 4x_2,\ \ x_2' = 3x_1 + 2x_2;\ \ x_1(0) = x_2(0) = 1$
4. $x_1' = 4x_1 + x_2,\ \ x_2' = 6x_1 - x_2$
5. $x_1' = 6x_1 - 7x_2,\ \ x_2' = x_1 - 2x_2$
6. $x_1' = 9x_1 + 5x_2,\ \ x_2' = -6x_1 - 2x_2;\ x_1(0) = 1,\ \ x_2(0) = 0$
7. $x_1' = -3x_1 + 4x_2,\ \ x_2' = 6x_1 - 5x_2$
8. $x_1' = x_1 - 5x_2,\ \ x_2' = x_1 - x_2$
9. $x_1' = 2x_1 - 5x_2,\ \ x_2' = 4x_1 - 2x_2;\ x_1(0) = 2,\ \ x_2(0) = 3$
10. $x_1' = -3x_1 - 2x_2,\ \ x_2' = 9x_1 + 3x_2$
11. $x_1' = x_1 - 2x_2,\ \ x_2' = 2x_1 + x_2;\ x_1(0) = 0,\ \ x_2(0) = 4$
12. $x_1' = x_1 - 5x_2,\ \ x_2' = x_1 + 3x_2$
13. $x_1' = 5x_1 - 9x_2,\ \ x_2' = 2x_1 - x_2$
14. $x_1' = 3x_1 - 4x_2,\ \ x_2' = 4x_1 + 3x_2$
15. $x_1' = 7x_1 - 5x_2,\ \ x_2' = 4x_1 + 3x_2$
16. $x_1' = -50x_1 + 20x_2,\ \ x_2' = 100x_1 - 60x_2$

In Problems 17 through 25, the eigenvalues of the coefficient matrix can be found by inspection and factoring. Apply the eigenvalue method to find a general solution of each system.

17. $x_1' = 4x_1 + x_2 + 4x_3,\ \ x_2' = x_1 + 7x_2 + x_3,$
$x_3' = 4x_1 + x_2 + 4x_3$
18. $x_1' = x_1 + 2x_2 + 2x_3,\ \ x_2' = 2x_1 + 7x_2 + x_3,$
$x_3' = 2x_1 + x_2 + 7x_3$
19. $x_1' = 4x_1 + x_2 + x_3,\ \ x_2' = x_1 + 4x_2 + x_3,\ \ x_3' = x_1 + x_2 + 4x_3$
20. $x_1' = 5x_1 + x_2 + 3x_3,\ \ x_2' = x_1 + 7x_2 + x_3,$
$x_3' = 3x_1 + x_2 + 5x_3$
21. $x_1' = 5x_1 - 6x_3,\ \ x_2' = 2x_1 - x_2 - 2x_3,\ \ x_3' = 4x_1 - 2x_2 - 4x_3$
22. $x_1' = 3x_1 + 2x_2 + 2x_3,\ \ x_2' = -5x_1 - 4x_2 - 2x_3,$
$x_3' = 5x_1 + 5x_2 + 3x_3$
23. $x_1' = 3x_1 + x_2 + x_3,\ \ x_2' = -5x_1 - 3x_2 - x_3,$
$x_3' = 5x_1 + 5x_2 + 3x_3$
24. $x_1' = 2x_1 + x_2 - x_3,\ \ x_2' = -4x_1 - 3x_2 - x_3,$
$x_3' = 4x_1 + 4x_2 + 2x_3$
25. $x_1' = 5x_1 + 5x_2 + 2x_3,\ \ x_2' = -6x_1 - 6x_2 - 5x_3,$
$x_3' = 6x_1 + 6x_2 + 5x_3$
26. Find the particular solution of the system

$$\frac{dx_1}{dt} = 3x_1 \qquad + x_3,$$
$$\frac{dx_2}{dt} = 9x_1 - x_2 + 2x_3,$$
$$\frac{dx_3}{dt} = -9x_1 + 4x_2 - x_3$$

that satisfies the initial conditions $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 17$.

The amounts $x_1(t)$ and $x_2(t)$ of salt in the two brine t
Fig. 5.2.7 satisfy the differential equations

$$\frac{dx_1}{dt} = -k_1 x_1, \qquad \frac{dx_2}{dt} = k_1 x_1 - k_2 x_2,$$

where $k_i = r/V_i$ for $i = 1, 2$. In Problems 27 and 28 t
umes V_1 and V_2 are given. First solve for $x_1(t)$ and $x_2($
suming that $r = 10$ (gal/min), $x_1(0) = 15$ (lb), and $x_2($
Then find the maximum amount of salt ever in tank 2. F
construct a figure showing the graphs of $x_1(t)$ and $x_2(t)$

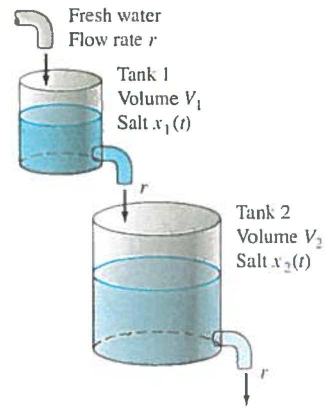


FIGURE 5.2.7. The two brine tanks of Problems 27 and 28.

27. $V_1 = 50$ (gal), $V_2 = 25$ (gal)
28. $V_1 = 25$ (gal), $V_2 = 40$ (gal)

The amounts $x_1(t)$ and $x_2(t)$ of salt in the two brine ta
Fig. 5.2.8 satisfy the differential equations

$$\frac{dx_1}{dt} = -k_1 x_1 + k_2 x_2, \qquad \frac{dx_2}{dt} = k_1 x_1 - k_2 x_2,$$

where $k_i = r/V_i$ as usual. In Problems 29 and 30, sol
$x_1(t)$ and $x_2(t)$, assuming that $r = 10$ (gal/min), $x_1(0)$
(lb), and $x_2(0) = 0$. Then construct a figure showi
graphs of $x_1(t)$ and $x_2(t)$.

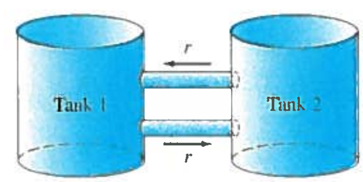


FIGURE 5.2.8. The two brine tanks of Problems 29 and 30.

29. $V_1 = 50$ (gal), $V_2 = 25$ (gal)
30. $V_1 = 25$ (gal), $V_2 = 40$ (gal)

Problems 31 through 34 deal with the open three-tank system of Fig. 5.2.2. Fresh water flows into tank 1; mixed brine flows from tank 1 into tank 2, from tank 2 into tank 3, and out of tank 3; all at the given flow rate r gallons per minute. The initial amounts $x_1(0) = x_0$ (lb), $x_2(0) = 0$, and $x_3(0) = 0$ of salt in the three tanks are given, as are their volumes V_1, V_2, and V_3 (in gallons). First solve for the amounts of salt in the three tanks at time t, then determine the maximal amount of salt that tank 3 ever contains. Finally, construct a figure showing the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

31. $r = 30$, $x_0 = 27$, $V_1 = 30$, $V_2 = 15$, $V_3 = 10$

32. $r = 60$, $x_0 = 45$, $V_1 = 20$, $V_2 = 30$, $V_3 = 60$

33. $r = 60$, $x_0 = 45$, $V_1 = 15$, $V_2 = 10$, $V_3 = 30$

34. $r = 60$, $x_0 = 40$, $V_1 = 20$, $V_2 = 12$, $V_3 = 60$

Problems 35 through 37 deal with the closed three-tank system of Fig. 5.2.5, which is described by the equations in (24). Mixed brine flows from tank 1 into tank 2, from tank 2 into tank 3, and from tank 3 into tank 1, all at the given flow rate r gallons per minute. The initial amounts $x_1(0) = x_0$ (pounds), $x_2(0) = 0$, and $x_3(0) = 0$ of salt in the three tanks are given, as are their volumes V_1, V_2, and V_3 (in gallons). First solve for the amounts of salt in the three tanks at time t, then determine the limiting amount (as $t \to +\infty$) of salt in each tank. Finally, construct a figure showing the graphs of $x_1(t)$, $x_2(t)$, and $x_3(t)$.

35. $r = 120$, $x_0 = 33$, $V_1 = 20$, $V_2 = 6$, $V_3 = 40$

36. $r = 10$, $x_0 = 18$, $V_1 = 20$, $V_2 = 50$, $V_3 = 20$

37. $r = 60$, $x_0 = 55$, $V_1 = 60$, $V_2 = 20$, $V_3 = 30$

For each matrix **A** *given in Problems 38 through 40, the zeros in the matrix make its characteristic polynomial easy to calculate. Find the general solution of* $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

38. $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 4 & 4 \end{bmatrix}$

39. $\mathbf{A} = \begin{bmatrix} -2 & 0 & 0 & 9 \\ 4 & 2 & 0 & -10 \\ 0 & 0 & -1 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

40. $\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ -21 & -5 & -27 & -9 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & -21 & -2 \end{bmatrix}$

41. The coefficient matrix **A** of the 4×4 system

$$\begin{aligned} x_1' &= 4x_1 + x_2 + x_3 + 7x_4, \\ x_2' &= x_1 + 4x_2 + 10x_3 + x_4, \\ x_3' &= x_1 + 10x_2 + 4x_3 + x_4, \\ x_4' &= 7x_1 + x_2 + x_3 + 4x_4 \end{aligned}$$

has eigenvalues $\lambda_1 = -3$, $\lambda_2 = -6$, $\lambda_3 = 10$, and $\lambda_4 = 15$. Find the particular solution of this system that satisfies the initial conditions

$$x_1(0) = 3, \quad x_2(0) = x_3(0) = 1, \quad x_4(0) = 3.$$

In Problems 42 through 50, use a calculator or computer system to calculate the eigenvalues and eigenvectors (as illustrated in the 5.2 Application below) in order to find a general solution of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ *with the given coefficient matrix* **A**.

42. $\mathbf{A} = \begin{bmatrix} -40 & -12 & 54 \\ 35 & 13 & -46 \\ -25 & -7 & 34 \end{bmatrix}$

43. $\mathbf{A} = \begin{bmatrix} -20 & 11 & 13 \\ 12 & -1 & -7 \\ -48 & 21 & 31 \end{bmatrix}$

44. $\mathbf{A} = \begin{bmatrix} 147 & 23 & -202 \\ -90 & -9 & 129 \\ 90 & 15 & -123 \end{bmatrix}$

45. $\mathbf{A} = \begin{bmatrix} 9 & -7 & -5 & 0 \\ -12 & 7 & 11 & 9 \\ 24 & -17 & -19 & -9 \\ -18 & 13 & 17 & 9 \end{bmatrix}$

46. $\mathbf{A} = \begin{bmatrix} 13 & -42 & 106 & 139 \\ 2 & -16 & 52 & 70 \\ 1 & 6 & -20 & -31 \\ -1 & -6 & 22 & 33 \end{bmatrix}$

47. $\mathbf{A} = \begin{bmatrix} 23 & -18 & -16 & 0 \\ -8 & 6 & 7 & 9 \\ 34 & -27 & -26 & -9 \\ -26 & 21 & 25 & 12 \end{bmatrix}$

48. $\mathbf{A} = \begin{bmatrix} 47 & -8 & 5 & -5 \\ -10 & 32 & 18 & -2 \\ 139 & -40 & -167 & -121 \\ -232 & 64 & 360 & 248 \end{bmatrix}$

49. $\mathbf{A} = \begin{bmatrix} 139 & -14 & -52 & -14 & 28 \\ -22 & 5 & 7 & 8 & -7 \\ 370 & -38 & -139 & -38 & 76 \\ 152 & -16 & -59 & -13 & 35 \\ 95 & -10 & -38 & -7 & 23 \end{bmatrix}$

50. $\mathbf{A} = \begin{bmatrix} 9 & 13 & 0 & 0 & 0 & -13 \\ -14 & 19 & -10 & -20 & 10 & 4 \\ -30 & 12 & -7 & -30 & 12 & 18 \\ -12 & 10 & -10 & -9 & 10 & 2 \\ 6 & 9 & 0 & 6 & 5 & -15 \\ -14 & 23 & -10 & -20 & 10 & 0 \end{bmatrix}$

Periodic and Transient Solutions

It follows from Theorem 4 of Section 5.1 that a particular solution of the forced system

$$\mathbf{x}'' = \mathbf{A}\mathbf{x} + \mathbf{F}_0 \cos \omega t \tag{36}$$

will be of the form

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t), \tag{37}$$

where $\mathbf{x}_p(t)$ is a particular solution of the nonhomogeneous system and $\mathbf{x}_c(t)$ is a solution of the corresponding homogeneous system. It is typical for the effects of frictional resistance in mechanical systems to damp out the complementary function solution $\mathbf{x}_c(t)$, so that

$$\mathbf{x}_c(t) \to \mathbf{0} \quad \text{as} \quad t \to +\infty. \tag{38}$$

Hence $\mathbf{x}_c(t)$ is a **transient solution** that depends only on the initial conditions; it dies out with time, leaving the **steady periodic solution** $\mathbf{x}_p(t)$ resulting from the external driving force:

$$\mathbf{x}(t) \to \mathbf{x}_p(t) \quad \text{as} \quad t \to +\infty. \tag{39}$$

As a practical matter, every physical system includes frictional resistance (however small) that damps out transient solutions in this manner.

5.3 Problems

Problems 1 through 7 deal with the mass-and-spring system shown in Fig. 5.3.11 with stiffness matrix

$$\mathbf{K} = \begin{bmatrix} -(k_1+k_2) & k_2 \\ k_2 & -(k_2+k_3) \end{bmatrix}$$

and with the given mks values for the masses and spring constants. Find the two natural frequencies of the system and describe its two natural modes of oscillation.

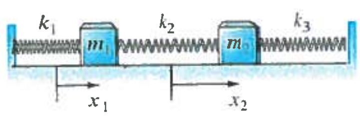


FIGURE 5.3.11. The mass-and-spring system for Problems 1 through 6.

1. $m_1 = m_2 = 1;\ \ k_1 = 0, k_2 = 2, k_3 = 0$ (no walls)

2. $m_1 = m_2 = 1;\ \ k_1 = 1, k_2 = 4, k_3 = 1$

3. $m_1 = 1, m_2 = 2;\ \ k_1 = 1, k_2 = k_3 = 2$

4. $m_1 = m_2 = 1;\ \ k_1 = 1, k_2 = 2, k_3 = 1$

5. $m_1 = m_2 = 1;\ \ k_1 = 2, k_2 = 1, k_3 = 2$

6. $m_1 = 1, m_2 = 2;\ \ k_1 = 2, k_2 = k_3 = 4$

7. $m_1 = m_2 = 1;\ \ k_1 = 4, k_2 = 6, k_3 = 4$

In Problems 8 through 10 the indicated mass-and-spring system is set in motion from rest ($x_1'(0) = x_2'(0) = 0$) in its equilibrium position ($x_1(0) = x_2(0) = 0$) with the given external forces $F_1(t)$ and $F_2(t)$ acting on the masses m_1 and m_2, respectively. Find the resulting motion of the system and describe it as a superposition of oscillations at three different frequencies.

8. The mass-and-spring system of Problem 2, with $F_1(t) = 96 \cos 5t$, $F_2(t) \equiv 0$

9. The mass-and-spring system of Problem 3, with $F_1(t) \equiv 0$, $F_2(t) = 120 \cos 3t$

10. The mass-and-spring system of Problem 7, with $F_1(t) = 30 \cos t$, $F_2(t) = 60 \cos t$

11. Consider a mass-and-spring system containing two masses $m_1 = 1$ and $m_2 = 1$ whose displacement functions $x(t)$ and $y(t)$ satisfy the differential equations

$$\begin{aligned} x'' &= -40x + 8y, \\ y'' &= 12x - 60y. \end{aligned}$$

(a) Describe the two fundamental modes of free oscillation of the system. **(b)** Assume that the two masses start in motion with the initial conditions

$$x(0) = 19, \quad x'(0) = 12$$

and

$$y(0) = 3, \quad y'(0) = 6$$

and are acted on by the same force, $F_1(t) = F_2(t) = -195 \cos 7t$. Describe the resulting motion as a superposition of oscillations at three different frequencies.

In Problems 12 and 13, find the natural frequencies of the three-mass system of Fig. 5.3.1, using the given masses and spring constants. For each natural frequency ω, give the ratio $a_1{:}a_2{:}a_3$ of amplitudes for a corresponding natural mode $x_1 = a_1 \cos \omega t$, $x_2 = a_2 \cos \omega t$, $x_3 = a_3 \cos \omega t$.

12. $m_1 = m_2 = m_3 = 1$; $k_1 = k_2 = k_3 = k_4 = 1$

13. $m_1 = m_2 = m_3 = 1$; $k_1 = k_2 = k_3 = k_4 = 2$ (*Hint*: One eigenvalue is $\lambda = -4$.)

14. In the system of Fig. 5.3.12, assume that $m_1 = 1$, $k_1 = 50$, $k_2 = 10$, and $F_0 = 5$ in mks units, and that $\omega = 10$. Then find m_2 so that in the resulting steady periodic oscillations, the mass m_1 will remain at rest(!). Thus the effect of the second mass-and-spring pair will be to neutralize the effect of the force on the first mass. This is an example of a *dynamic damper*. It has an electrical analogy that some cable companies use to prevent your reception of certain cable channels.

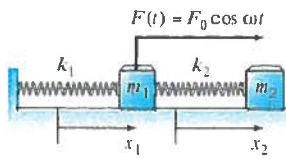


FIGURE 5.3.12. The mechanical system of Problem 14.

15. Suppose that $m_1 = 2$, $m_2 = \frac{1}{2}$, $k_1 = 75$, $k_2 = 25$, $F_0 = 100$, and $\omega = 10$ (all in mks units) in the forced mass-and-spring system of Fig. 5.3.9. Find the solution of the system $\mathbf{M}\mathbf{x}'' = \mathbf{K}\mathbf{x} + \mathbf{F}$ that satisfies the initial conditions $\mathbf{x}(0) = \mathbf{x}'(0) = \mathbf{0}$.

16. Figure 5.3.13 shows two railway cars with a buffer spring. We want to investigate the transfer of momentum that occurs after car 1 with initial velocity v_0 impacts car 2 at rest. The analog of Eq. (18) in the text is

$$\mathbf{x}'' = \begin{bmatrix} -c_1 & c_1 \\ c_2 & -c_2 \end{bmatrix} \mathbf{x}$$

with $c_i = k/m_i$ for $i = 1, 2$. Show that the eigenvalues of the coefficient matrix $\mathbf{A}$ are $\lambda_1 = 0$ and $\lambda_2 = -c_1 - c_2$, with associated eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $\mathbf{v}_2 = \begin{bmatrix} c_1 & -c_2 \end{bmatrix}^T$.

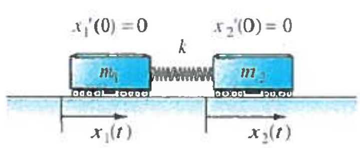


FIGURE 5.3.13. The two railway cars of Problems 16 through 19.

17. If the two cars of Problem 16 both weigh 16 tons (so that $m_1 = m_2 = 1000$ (slugs)) and $k = 1$ ton/ft (that is, 2000 lb/ft), show that the cars separate after $\pi/2$ seconds, and that $x_1'(t) = 0$ and $x_2'(t) = v_0$ thereafter. Thus the original momentum of car 1 is completely transferred to car 2.

18. If cars 1 and 2 weigh 8 and 16 tons, respectively, and $k = 3000$ lb/ft, show that the two cars separate after $\pi/3$ seconds, and that

$$x_1'(t) = -\tfrac{1}{3}v_0 \quad \text{and} \quad x_2'(t) = +\tfrac{2}{3}v_0$$

thereafter. Thus the two cars rebound in opposite directions.

19. If cars 1 and 2 weigh 24 and 8 tons, respectively, and $k = 1500$ lb/ft, show that the cars separate after $\pi/2$ seconds, and that

$$x_1'(t) = +\tfrac{1}{2}v_0 \quad \text{and} \quad x_2'(t) = +\tfrac{3}{2}v_0$$

thereafter. Thus both cars continue in the original direction of motion, but with different velocities.

Problems 20 through 23 deal with the same system of three railway cars (same masses) and two buffer springs (same spring constants) as shown in Fig. 5.3.6 and discussed in Example 2. The cars engage at time $t = 0$ with $x_1(0) = x_2(0) = x_3(0) = 0$ and with the given initial velocities (where $v_0 = 48$ ft/s). Show that the railway cars remain engaged until $t = \pi/2$ (s), after which time they proceed in their respective ways with constant velocities. Determine the values of these constant final velocities $x_1'(t)$, $x_2'(t)$, and $x_3'(t)$ of the three cars for $t > \pi/2$. In each problem you should find (as in Example 2) that the first and third railway cars exchange behaviors in some appropriate sense.

20. $x_1'(0) = v_0$, $x_2'(0) = 0$, $x_3'(0) = -v_0$

21. $x_1'(0) = 2v_0$, $x_2'(0) = 0$, $x_3'(0) = -v_0$

22. $x_1'(0) = v_0$, $x_2'(0) = v_0$, $x_3'(0) = -2v_0$

23. $x_1'(0) = 3v_0$, $x_2'(0) = 2v_0$, $x_3'(0) = 2v_0$

24. In the three-railway-car system of Fig. 5.3.6, suppose that cars 1 and 3 each weigh 32 tons, that car 2 weighs 8 tons, and that each spring constant is 4 tons/ft. If $x_1'(0) = v_0$ and $x_2'(0) = x_3'(0) = 0$, show that the two springs are compressed until $t = \pi/2$ and that

$$x_1'(t) = -\tfrac{1}{9}v_0 \quad \text{and} \quad x_2'(t) = x_3'(t) = +\tfrac{8}{9}v_0$$

thereafter. Thus car 1 rebounds, but cars 2 and 3 continue with the same velocity.

The Two-Axle Automobile

In Example 4 of Section 3.6 we investigated the vertical oscillations of a one-axle car—actually a unicycle. Now we can analyze a more realistic model: a car with two axles and with separate front and rear suspension systems. Figure 5.3.14 represents the suspension system of such a car. We assume that the car body acts as would a solid bar of mass m and length $L = L_1 + L_2$. It has moment of inertia I about its center of *mass C, which is at distance L_1 from the front of the car. The car has front and back suspension springs with Hooke's constants k_1 and k_2, respectively. When the car is in motion, let $x(t)$ denote the vertical displacement of the center of mass of the car from equilibrium; let $\theta(t)$ denote its angular displacement (in radians) from the horizontal. Then Newton's laws of motion for linear and angular acceleration can be used to derive the equations*

$$\begin{aligned} mx'' &= -(k_1 + k_2)x + (k_1 L_1 - k_2 L_2)\theta, \\ I\theta'' &= (k_1 L_1 - k_2 L_2)x - (k_1 L_1^2 + k_2 L_2^2)\theta. \end{aligned} \tag{40}$$

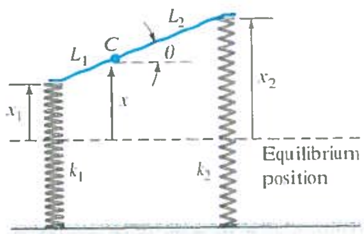


FIGURE 5.3.14. Model of the two-axle automobile.

25. Suppose that $m = 75$ slugs (the car weighs 2400 lb), $L_1 = 7$ ft, $L_2 = 3$ ft (it's a rear-engine car), $k_1 = k_2 = 2000$ lb/ft, and $I = 1000$ ft·lb·s². Then the equations in (40) take the form

$$\begin{aligned} 75x'' + 4000x - 8000\theta &= 0, \\ 1000\theta'' - 8000x + 116{,}000\theta &= 0. \end{aligned}$$

(a) Find the two natural frequencies ω_1 and ω_2 of the car. **(b)** Now suppose that the car is driven at a speed of v feet per second along a washboard surface shaped like a sine curve with a wavelength of 40 ft. The result is a periodic force on the car with frequency $\omega = 2\pi v/40 = \pi v/20$. Resonance occurs when with $\omega = \omega_1$ or $\omega = \omega_2$. Find the corresponding two critical speeds of the car (in feet per second and in miles per hour).

26. Suppose that $k_1 = k_2 = k$ and $L_1 = L_2 = \frac{1}{2}L$ in Fig. 5.3.14 (the symmetric situation). Then show that every free oscillation is a combination of a vertical oscillation with frequency

$$\omega_1 = \sqrt{2k/m}$$

and an angular oscillation with frequency

$$\omega_2 = \sqrt{kL^2/(2I)}.$$

*In Problems 27 through 29, the system of Fig. 5.3.14 is taken as a model for an undamped car with the given parameters in fps units. **(a)** Find the two natural frequencies of oscillation (in hertz). **(b)** Assume that this car is driven along a sinusoidal washboard surface with a wavelength of 40 ft. Find the two critical speeds.*

27. $m = 100$, $I = 800$, $L_1 = L_2 = 5$, $k_1 = k_2 = 2000$

28. $m = 100$, $I = 1000$, $L_1 = 6$, $L_2 = 4$, $k_1 = k_2 = 2000$

29. $m = 100$, $I = 800$, $L_1 = L_2 = 5$, $k_1 = 1000$, $k_2 = 2000$

5.3 Application Earthquake-Induced Vibrations of Multistory Buildings

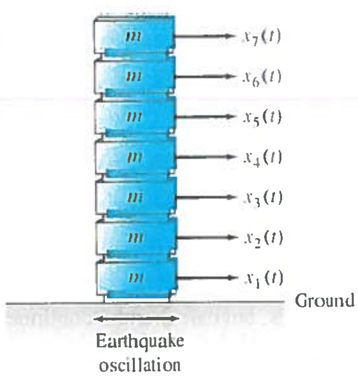


FIGURE 5.3.15. The seven-story building.

FIGURE 5.3.16. Forces on the ith floor.

In this application you are to investigate the response to transverse earthquake ground oscillations of the seven-story building illustrated in Fig. 5.3.15. Suppose that each of the seven (above-ground) floors weighs 16 tons, so the mass of each is $m = 1000$ (slugs). Also assume a horizontal restoring force of $k = 5$ (tons per foot) between adjacent floors. That is, the internal forces in response to horizontal displacements of the individual floors are those shown in Fig. 5.3.16. It follows that the free transverse oscillations indicated in Fig. 5.3.15 satisfy the equation $\mathbf{Mx}'' = \mathbf{Kx}$ with $n = 7$, $m_i = 1000$ (for each i), and $k_i = 10000$ (lb/ft) for $1 \leqq i \leqq 7$. The system then reduces to the form $\mathbf{x}'' = \mathbf{Ax}$ with

$$\mathbf{A} = \begin{bmatrix} -20 & 10 & 0 & 0 & 0 & 0 & 0 \\ 10 & -20 & 10 & 0 & 0 & 0 & 0 \\ 0 & 10 & -20 & 10 & 0 & 0 & 0 \\ 0 & 0 & 10 & -20 & 10 & 0 & 0 \\ 0 & 0 & 0 & 10 & -20 & 10 & 0 \\ 0 & 0 & 0 & 0 & 10 & -20 & 10 \\ 0 & 0 & 0 & 0 & 0 & 10 & -10 \end{bmatrix}. \tag{1}$$

Once the matrix $\mathbf{A}$ has been entered, the TI-86 command **eigVl A** takes about 15 seconds to calculate the seven eigenvalues shown in the λ-column of the table

For instance, suppose that $x_1(0) = x_2(0) = 0$ and that $x_1'(0) = x_2'(0) = v_0$. Then the equations

$$\begin{aligned}
x_1(0) &= c_1 + c_2 + c_3 &&= 0,\\
x_2(0) &= c_1 + c_2 - c_3 &&= 0,\\
x_1'(0) &= \quad -2c_2 - 2c_3 + c_4 &&= v_0,\\
x_2'(0) &= \quad -2c_2 + 2c_3 - c_4 &&= v_0
\end{aligned}$$

are readily solved for $c_1 = \frac{1}{2}v_0$, $c_2 = -\frac{1}{2}v_0$, and $c_3 = c_4 = 0$, so

$$\begin{aligned}
x_1(t) = x_2(t) &= \tfrac{1}{2}v_0\left(1 - e^{-2t}\right),\\
x_1'(t) = x_2'(t) &= v_0 e^{-2t}.
\end{aligned}$$

In this case the two railway cars continue in the same direction with equal but exponentially damped velocities, approaching the displacements $x_1 = x_2 = \frac{1}{2}v_0$ as $t \to +\infty$.

It is of interest to interpret physically the individual generalized eigenvector solutions given in (33). The degenerate ($\lambda_0 = 0$) solution

$$\mathbf{x}_1(t) = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$$

describes the two masses at rest with position functions $x_1(t) \equiv 1$ and $x_2(t) \equiv 1$. The solution

$$\mathbf{x}_2(t) = \begin{bmatrix} 1 & 1 & -2 & -2 \end{bmatrix}^T e^{-2t}$$

corresponding to the carefully chosen eigenvector $\mathbf{w}_1$ describes damped motions $x_1(t) = e^{-2t}$ and $x_2(t) = e^{-2t}$ of the two masses, with equal velocities in the same direction. Finally, the solutions $\mathbf{x}_3(t)$ and $\mathbf{x}_4(t)$ resulting from the length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ both describe damped motion with the two masses moving in opposite directions. ■

The methods of this section apply to complex multiple eigenvalues just as to real multiple eigenvalues (although the necessary computations tend to be somewhat lengthy). Given a complex conjugate pair $\alpha \pm \beta i$ of eigenvalues of multiplicity k, we work with one of them (say, $\alpha - \beta i$) as if it were real to find k independent complex-valued solutions. The real and imaginary parts of these complex-valued solutions then provide $2k$ real-valued solutions associated with the two eigenvalues $\lambda = \alpha - \beta i$ and $\bar{\lambda} = \alpha + \beta i$ each of multiplicity k. See Problems 33 and 34.

5.4 Problems

Find general solutions of the systems in Problems 1 through 22. In Problems 1 through 6, use a computer system or graphing calculator to construct a direction field and typical solution curves for the given system.

1. $\mathbf{x}' = \begin{bmatrix} -2 & 1 \\ -1 & -4 \end{bmatrix}\mathbf{x}$

2. $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{x}$

3. $\mathbf{x}' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}\mathbf{x}$

4. $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix}\mathbf{x}$

5. $\mathbf{x}' = \begin{bmatrix} 7 & 1 \\ -4 & 3 \end{bmatrix}\mathbf{x}$

6. $\mathbf{x}' = \begin{bmatrix} 1 & -4 \\ 4 & 9 \end{bmatrix}\mathbf{x}$

7. $\mathbf{x}' = \begin{bmatrix} 2 & 0 & 0 \\ -7 & 9 & 7 \\ 0 & 0 & 2 \end{bmatrix}\mathbf{x}$

8. $\mathbf{x}' = \begin{bmatrix} 25 & 12 & 0 \\ -18 & -5 & 0 \\ 6 & 6 & 13 \end{bmatrix}\mathbf{x}$

9. $\mathbf{x}' = \begin{bmatrix} -19 & 12 & 84 \\ 0 & 5 & 0 \\ -8 & 4 & 33 \end{bmatrix}\mathbf{x}$

10. $\mathbf{x}' = \begin{bmatrix} -13 & 40 & -48 \\ -8 & 23 & -24 \\ 0 & 0 & 3 \end{bmatrix}\mathbf{x}$

11. $\mathbf{x}' = \begin{bmatrix} -3 & 0 & -4 \\ -1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix}\mathbf{x}$

12. $\mathbf{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\mathbf{x}$

13. $\mathbf{x}' = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & -4 \\ 0 & 1 & -3 \end{bmatrix}\mathbf{x}$

14. $\mathbf{x}' = \begin{bmatrix} 0 & 0 & 1 \\ -5 & -1 & -5 \\ 4 & 1 & -2 \end{bmatrix}\mathbf{x}$

15. $\mathbf{x}' = \begin{bmatrix} -2 & -9 & 0 \\ 1 & 4 & 0 \\ 1 & 3 & 1 \end{bmatrix}\mathbf{x}$

16. $\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ -2 & -2 & -3 \\ 2 & 3 & 4 \end{bmatrix}\mathbf{x}$

17. $\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 18 & 7 & 4 \\ -27 & -9 & -5 \end{bmatrix}\mathbf{x}$

18. $\mathbf{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 1 \\ -2 & -4 & -1 \end{bmatrix}\mathbf{x}$

19. $\mathbf{x}' = \begin{bmatrix} 1 & -4 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 6 & -12 & -1 & -6 \\ 0 & -4 & 0 & -1 \end{bmatrix}\mathbf{x}$

20. $\mathbf{x}' = \begin{bmatrix} 2 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}\mathbf{x}$

21. $\mathbf{x}' = \begin{bmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}\mathbf{x}$

22. $\mathbf{x}' = \begin{bmatrix} 1 & 3 & 7 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & -6 & -14 & 1 \end{bmatrix}\mathbf{x}$

In Problems 23 through 32 the eigenvalues of the coefficient matrix $\mathbf{A}$ *are given. Find a general solution of the indicated system* $\mathbf{x}' = \mathbf{A}\mathbf{x}$. *Especially in Problems 29 through 32, use of a computer algebra system (as in the application material for this section) may be useful.*

23. $\mathbf{x}' = \begin{bmatrix} 39 & 8 & -16 \\ -36 & -5 & 16 \\ 72 & 16 & -29 \end{bmatrix}\mathbf{x}$; $\lambda = -1, 3, 3$

24. $\mathbf{x}' = \begin{bmatrix} 28 & 50 & 100 \\ 15 & 33 & 60 \\ -15 & -30 & -57 \end{bmatrix}\mathbf{x}$; $\lambda = -2, 3, 3$

25. $\mathbf{x}' = \begin{bmatrix} -2 & 17 & 4 \\ -1 & 6 & 1 \\ 0 & 1 & 2 \end{bmatrix}\mathbf{x}$; $\lambda = 2, 2, 2$

26. $\mathbf{x}' = \begin{bmatrix} 5 & -1 & 1 \\ 1 & 3 & 0 \\ -3 & 2 & 1 \end{bmatrix}\mathbf{x}$; $\lambda = 3, 3, 3$

27. $\mathbf{x}' = \begin{bmatrix} -3 & 5 & -5 \\ 3 & -1 & 3 \\ 8 & -8 & 10 \end{bmatrix}\mathbf{x}$; $\lambda = 2, 2, 2$

28. $\mathbf{x}' = \begin{bmatrix} -15 & -7 & 4 \\ 34 & 16 & -11 \\ 17 & 7 & 5 \end{bmatrix}\mathbf{x}$; $\lambda = 2, 2, 2$

29. $\mathbf{x}' = \begin{bmatrix} -1 & 1 & 1 & -2 \\ 7 & -4 & -6 & 11 \\ 5 & -1 & 1 & 3 \\ 6 & -2 & -2 & 6 \end{bmatrix}\mathbf{x}$; $\lambda = -1, -1, 2, 2$

30. $\mathbf{x}' = \begin{bmatrix} 2 & 1 & -2 & 1 \\ 0 & 3 & -5 & 3 \\ 0 & -13 & 22 & -12 \\ 0 & -27 & 45 & -25 \end{bmatrix}\mathbf{x}$; $\lambda = -1, -1, 2, 2$

31. $\mathbf{x}' = \begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix}\mathbf{x}$; $\lambda = 1, 1, 1, 1$

32. $\mathbf{x}' = \begin{bmatrix} 11 & -1 & 26 & 6 & -3 \\ 0 & 3 & 0 & 0 & 0 \\ -9 & 0 & -24 & -6 & 3 \\ 3 & 0 & 9 & 5 & -1 \\ -48 & -3 & -138 & -30 & 18 \end{bmatrix}\mathbf{x}$;
$\lambda = 2, 2, 3, 3, 3$

33. The characteristic equation of the coefficient matrix $\mathbf{A}$ of the system

$$\mathbf{x}' = \begin{bmatrix} 3 & -4 & 1 & 0 \\ 4 & 3 & 0 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{bmatrix}\mathbf{x}$$

is

$$\phi(\lambda) = (\lambda^2 - 6\lambda + 25)^2 = 0.$$

Therefore, $\mathbf{A}$ has the repeated complex conjugate pair $3 \pm 4i$ of eigenvalues. First show that the complex vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 & i & 0 & 0 \end{bmatrix}^T \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 9 & 0 & 1 & i \end{bmatrix}^T$$

form a length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ associated with the eigenvalue $\lambda = 3 - 4i$. Then calculate the real and imaginary parts of the complex-valued solutions

$$\mathbf{v}_1 e^{\lambda t} \quad \text{and} \quad (\mathbf{v}_1 t + \mathbf{v}_2)e^{\lambda t}$$

to find four independent real-valued solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

34. The characteristic equation of the coefficient matrix $\mathbf{A}$ of the system

$$\mathbf{x}' = \begin{bmatrix} 2 & 0 & -8 & -3 \\ -18 & -1 & 0 & 0 \\ -9 & -3 & -25 & -9 \\ 33 & 10 & 90 & 32 \end{bmatrix} \mathbf{x}$$

is

$$\phi(\lambda) = (\lambda^2 - 4\lambda + 13)^2 = 0.$$

Therefore, $\mathbf{A}$ has the repeated complex conjugate pair $2 \pm 3i$ of eigenvalues. First show that the complex vectors

$$\mathbf{v}_1 = \begin{bmatrix} -i & 3+3i & 0 & -i \end{bmatrix}^T,$$

$$\mathbf{v}_2 = \begin{bmatrix} 3 & -10+9i & -i & 0 \end{bmatrix}^T$$

form a length 2 chain $\{\mathbf{v}_1, \mathbf{v}_2\}$ associated with the eigenvalue $\lambda = 2 + 3i$. Then calculate (as in Problem 33) four independent real-valued solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

35. Find the position functions $x_1(t)$ and $x_2(t)$ of the railway cars of Fig. 5.4.1 if the physical parameters are given by

$$m_1 = m_2 = c_1 = c_2 = c = k = 1$$

and the initial conditions are

$$x_1(0) = x_2(0) = 0, \quad x_1'(0) = x_2'(0) = v_0.$$

How far do the cars travel before stopping?

36. Repeat Problem 35 under the assumption that car 1 is shielded from air resistance by car 2, so now $c_1 = 0$. Show that, before stopping, the cars travel twice as far as those of Problem 35.

5.4 Application Defective Eigenvalues and Generalized Eigenvectors

A typical computer algebra system can calculate both the eigenvalues of a given matrix $\mathbf{A}$ and the linearly independent (ordinary) eigenvectors associated with each eigenvalue. For instance, consider the 4×4 matrix

$$\mathbf{A} = \begin{bmatrix} 35 & -12 & 4 & 30 \\ 22 & -8 & 3 & 19 \\ -10 & 3 & 0 & -9 \\ -27 & 9 & -3 & -23 \end{bmatrix} \tag{1}$$

of Problem 31 in this section. When the matrix $\mathbf{A}$ has been entered, the *Maple* calculation

```
with(linalg): eigenvectors(A);
[1, 4, {[-1, 0, 1, 1], [0, 1, 3, 0]}]
```

or the *Mathematica* calculation

```
Eigensystem[A]
{{1,1,1,1},
{{-3,-1,0,3}, {0,1,3,0}, {0,0,0,0}, {0,0,0,0}}}
```

reveals that the matrix $\mathbf{A}$ in Eq. (1) has the single eigenvalue $\lambda = 1$ of multiplicity 4 with only two independent associated eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. The MATLAB command

```
[V, D] = eig(sym(A))
```

provides the same information. The eigenvalue $\lambda = 1$ therefore has defect $d = 2$. If $\mathbf{B} = \mathbf{A} - (1)\mathbf{I}$, you should find that $\mathbf{B}^2 \neq \mathbf{0}$ but $\mathbf{B}^3 = \mathbf{0}$. If

$$\mathbf{u}_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T, \quad \mathbf{u}_2 = \mathbf{B}\mathbf{u}_1, \quad \mathbf{u}_3 = \mathbf{B}\mathbf{u}_2,$$

then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ should be a length 3 chain of generalized eigenvectors based on the ordinary eigenvector $\mathbf{u}_3$ (which should be a linear combination of the original eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$). Use your computer algebra system to carry out this construction, and finally write four linearly independent solutions of the linear system $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

The first two equations $8b + 16c = 0$ and $4b + 8c = 0$ are satisfied by $b = 2$ and $c = -1$, but leave a arbitrary. With $a = 0$ we get the generalized eigenvector $\mathbf{u}_3 = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix}^T$ of rank $r = 2$ associated with the eigenvalue $\lambda = 3$. Because $(\mathbf{A} - 3\mathbf{I})^2\mathbf{u} = \mathbf{0}$, Eq. (34) yields the third solution

$$\mathbf{x}_3(t) = e^{3t}\left[\mathbf{u}_3 + (\mathbf{A} - 3\mathbf{I})\mathbf{u}_3 t\right]$$
$$= e^{3t}\left(\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 0 & 4 & 5 \\ 0 & 2 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} t\right) = e^{3t}\begin{bmatrix} 3t \\ 2 \\ -1 \end{bmatrix}. \tag{40}$$

With the solutions listed in Eqs. (39) and (40), the fundamental matrix

$$\mathbf{\Phi}(t) = \begin{bmatrix} \mathbf{x}_1(t) & \mathbf{x}_2(t) & \mathbf{x}_3(t) \end{bmatrix}$$

defined by Eq. (35) is

$$\mathbf{\Phi}(t) = \begin{bmatrix} 2e^{5t} & e^{3t} & 3te^{3t} \\ e^{5t} & 0 & 2e^{3t} \\ 0 & 0 & -e^{3t} \end{bmatrix} \quad \text{with} \quad \mathbf{\Phi}(0)^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & -4 \\ 0 & 0 & -1 \end{bmatrix}.$$

Hence Theorem 3 finally yields

$$e^{\mathbf{A}t} = \mathbf{\Phi}(t)\mathbf{\Phi}(0)^{-1}$$
$$= \begin{bmatrix} 2e^{5t} & e^{3t} & 3te^{3t} \\ e^{5t} & 0 & 2e^{3t} \\ 0 & 0 & -e^{3t} \end{bmatrix}\begin{bmatrix} 0 & 1 & 2 \\ 1 & -2 & -4 \\ 0 & 0 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} e^{3t} & 2e^{5t} - 2e^{3t} & 4e^{5t} - (4+3t)e^{3t} \\ 0 & e^{5t} & 2e^{5t} - 2e^{3t} \\ 0 & 0 & e^{3t} \end{bmatrix}.$$

Remark: As in Example 7, Theorem 3 suffices for the computation of $e^{\mathbf{A}t}$ provided that a basis consisting of generalized eigenvectors of $\mathbf{A}$ can be found. Alternatively, a computer algebra system can be used as indicated in the project material for this section.

5.5 Problems

Find a fundamental matrix of each of the systems in Problems 1 through 8, then apply Eq. (8) to find a solution satisfying the given initial conditions.

1. $\mathbf{x}' = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$

2. $\mathbf{x}' = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

3. $\mathbf{x}' = \begin{bmatrix} 2 & -5 \\ 4 & -2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

4. $\mathbf{x}' = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

5. $\mathbf{x}' = \begin{bmatrix} -3 & -2 \\ 9 & 3 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

6. $\mathbf{x}' = \begin{bmatrix} 7 & -5 \\ 4 & 3 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

7. $\mathbf{x}' = \begin{bmatrix} 5 & 0 & -6 \\ 2 & -1 & -2 \\ 4 & -2 & -4 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

8. $\mathbf{x}' = \begin{bmatrix} 3 & 2 & 2 \\ -5 & -4 & -2 \\ 5 & 5 & 3 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

Compute the matrix exponential $e^{\mathbf{A}t}$ for each system $\mathbf{x}' = \mathbf{A}\mathbf{x}$ given in Problems 9 through 20.

9. $x_1' = 5x_1 - 4x_2,\ x_2' = 2x_1 - x_2$
10. $x_1' = 6x_1 - 6x_2,\ x_2' = 4x_1 - 4x_2$
11. $x_1' = 5x_1 - 3x_2,\ x_2' = 2x_1$
12. $x_1' = 5x_1 - 4x_2,\ x_2' = 3x_1 - 2x_2$
13. $x_1' = 9x_1 - 8x_2,\ x_2' = 6x_1 - 5x_2$
14. $x_1' = 10x_1 - 6x_2,\ x_2' = 12x_1 - 7x_2$
15. $x_1' = 6x_1 - 10x_2,\ x_2' = 2x_1 - 3x_2$
16. $x_1' = 11x_1 - 15x_2,\ x_2' = 6x_1 - 8x_2$
17. $x_1' = 3x_1 + x_2,\ x_2' = x_1 + 3x_2$
18. $x_1' = 4x_1 + 2x_2,\ x_2' = 2x_1 + 4x_2$
19. $x_1' = 9x_1 + 2x_2,\ x_2' = 2x_1 + 6x_2$
20. $x_1' = 13x_1 + 4x_2,\ x_2' = 4x_1 + 7x_2$

In Problems 21 through 24, show that the matrix A is nilpotent and then use this fact to find (as in Example 3) the matrix exponential $e^{\mathbf{A}t}$.

21. $\mathbf{A} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ **22.** $\mathbf{A} = \begin{bmatrix} 6 & 4 \\ -9 & -6 \end{bmatrix}$

23. $\mathbf{A} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ **24.** $\mathbf{A} = \begin{bmatrix} 3 & 0 & -3 \\ 5 & 0 & 7 \\ 3 & 0 & -3 \end{bmatrix}$

Each coefficient matrix **A** *in Problems 25 through 30 is the sum of a nilpotent matrix and a multiple of the identity matrix. Use this fact (as in Example 6) to solve the given initial value problem.*

25. $\mathbf{x}' = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

26. $\mathbf{x}' = \begin{bmatrix} 7 & 0 \\ 11 & 7 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$

27. $\mathbf{x}' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

28. $\mathbf{x}' = \begin{bmatrix} 5 & 0 & 0 \\ 10 & 5 & 0 \\ 20 & 30 & 5 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 40 \\ 50 \\ 60 \end{bmatrix}$

29. $\mathbf{x}' = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 6 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

30. $\mathbf{x}' = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 6 & 3 & 0 & 0 \\ 9 & 6 & 3 & 0 \\ 12 & 9 & 6 & 3 \end{bmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

31. Suppose that the $n \times n$ matrices **A** and **B** commute; that is, that $\mathbf{AB} = \mathbf{BA}$. Prove that $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}}$. (*Suggestion*: Group the terms in the product of the two series on the right-hand side to obtain the series on the left.)

32. Deduce from the result of Problem 31 that, for every square matrix **A**, the matrix $e^{\mathbf{A}}$ is nonsingular with $\left(e^{\mathbf{A}}\right)^{-1} = e^{-\mathbf{A}}$.

33. Suppose that

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Show that $\mathbf{A}^{2n} = \mathbf{I}$ and that $\mathbf{A}^{2n+1} = \mathbf{A}$ if n is a positive integer. Conclude that

$$e^{\mathbf{A}t} = \mathbf{I}\cosh t + \mathbf{A}\sinh t,$$

and apply this fact to find a general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$. Verify that it is equivalent to the general solution found by the eigenvalue method.

34. Suppose that

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}.$$

Show that $e^{\mathbf{A}t} = \mathbf{I}\cos 2t + \frac{1}{2}\mathbf{A}\sin 2t$. Apply this fact to find a general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and verify that it is equivalent to the solution found by the eigenvalue method.

Apply Theorem 3 to calculate the matrix exponential $e^{\mathbf{A}t}$ for each of the matrices in Problems 35 through 40.

35. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 0 & 3 \end{bmatrix}$ **36.** $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$

37. $\mathbf{A} = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ **38.** $\mathbf{A} = \begin{bmatrix} 5 & 20 & 30 \\ 0 & 10 & 20 \\ 0 & 0 & 5 \end{bmatrix}$

39. $\mathbf{A} = \begin{bmatrix} 1 & 3 & 3 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ **40.** $\mathbf{A} = \begin{bmatrix} 2 & 4 & 4 & 4 \\ 0 & 2 & 4 & 4 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix}$

5.5 Application Automated Matrix Exponential Solutions

If **A** is an $n \times n$ matrix, then a computer algebra system can be used first to calculate the fundamental matrix $e^{\mathbf{A}t}$ for the system $\mathbf{x}' = \mathbf{A}\mathbf{x}$, then to calculate the matrix product $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_0$ to obtain a solution satisfying the initial condition $\mathbf{x}(0) = \mathbf{x}_0$. For instance, suppose that we want to solve the initial value problem

$$\begin{aligned} x_1' &= 13x_1 + 4x_2, \\ x_2' &= 4x_1 + 7x_2; \\ x_1(0) &= 11, \quad x_2(0) = 23. \end{aligned}$$

The first component of this column vector is

$$y_p = \begin{bmatrix} y_1 & y_2 \end{bmatrix} \int \frac{1}{W} \begin{bmatrix} -y_2 f \\ y_1 f \end{bmatrix} dt = -y_1 \int \frac{y_2 f}{W}\, dt + y_2 \int \frac{y_1 f}{W}\, dt.$$

If, finally, we supply the independent variable t throughout, the final result on the right-hand side here is simply the variation of parameters formula in Eq. (33) of Section 3.5 (where, however, the independent variable is denoted by x).

5.6 Problems

Apply the method of undetermined coefficients to find a particular solution of each of the systems in Problems 1 through 14. If initial conditions are given, find the particular solution that satisfies these conditions. Primes denote derivatives with respect to t.

1. $x' = x + 2y + 3,\ y' = 2x + y - 2$
2. $x' = 2x + 3y + 5,\ y' = 2x + y - 2t$
3. $x' = 3x + 4y,\ y' = 3x + 2y + t^2;\ x(0) = y(0) = 0$
4. $x' = 4x + y + e^t,\ y' = 6x - y - e^t;\ x(0) = y(0) = 1$
5. $x' = 6x - 7y + 10,\ y' = x - 2y - 2e^{-t}$
6. $x' = 9x + y + 2e^t,\ y' = -8x - 2y + te^t$
7. $x' = -3x + 4y + \sin t,\ y' = 6x - 5y;\ x(0) = 1,\ y(0) = 0$
8. $x' = x - 5y + 2\sin t,\ y' = x - y - 3\cos t$
9. $x' = x - 5y + \cos 2t,\ y' = x - y$
10. $x' = x - 2y,\ y' = 2x - y + e^t \sin t$
11. $x' = 2x + 4y + 2,\ y' = x + 2y + 3;\ x(0) = 1,\ y(0) = -1$
12. $x' = x + y + 2t,\ y' = x + y - 2t$
13. $x' = 2x + y + 2e^t,\ y' = x + 2y - 3e^t$
14. $x' = 2x + y + 1,\ y' = 4x + 2y + e^{4t}$

Problems 15 and 16 are similar to Example 2, but with two brine tanks (having volumes V_1 and V_2 gallons as in Fig. 5.6.2) instead of three tanks. Each tank initially contains fresh water, and the inflow to tank 1 at the rate of r gallons per minute has a salt concentration of c_0 pounds per gallon. ***(a)*** *Find the amounts $x_1(t)$ and $x_2(t)$ of salt in the two tanks after t minutes.* ***(b)*** *Find the limiting (long-term) amount of salt in each tank.* ***(c)*** *Find how long it takes for each tank to reach a salt concentration of 1 lb/gal.*

15. $V_1 = 100,\ V_2 = 200,\ r = 10,\ c_0 = 2$
16. $V_1 = 200,\ V_2 = 100,\ r = 10,\ c_0 = 3$

In Problems 17 through 34, use the method of variation of parameters (and perhaps a computer algebra system) to solve the initial value problem

$$\mathbf{x}' = \mathbf{A}\mathbf{x} + \mathbf{f}(t), \quad \mathbf{x}(a) = \mathbf{x}_a.$$

In each problem we provide the matrix exponential $e^{\mathbf{A}t}$ as provided by a computer algebra system.

17. $\mathbf{A} = \begin{bmatrix} 6 & -7 \\ 1 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 60 \\ 90 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

$$e^{\mathbf{A}t} = \frac{1}{6}\begin{bmatrix} -e^{-t} + 7e^{5t} & 7e^{-t} - 7e^{5t} \\ -e^{-t} + e^{5t} & 7e^{-t} - e^{5t} \end{bmatrix}$$

18. Repeat Problem 17, but with $\mathbf{f}(t)$ replaced with $\begin{bmatrix} 100t \\ -50t \end{bmatrix}$.

19. $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 180t \\ 90 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

$$e^{\mathbf{A}t} = \frac{1}{5}\begin{bmatrix} e^{-3t} + 4e^{2t} & -2e^{-3t} + 2e^{2t} \\ -2e^{-3t} + 2e^{2t} & 4e^{-3t} + e^{2t} \end{bmatrix}$$

20. Repeat Problem 19, but with $\mathbf{f}(t)$ replaced with $\begin{bmatrix} 75e^{2t} \\ 0 \end{bmatrix}$.

21. $\mathbf{A} = \begin{bmatrix} 4 & -1 \\ 5 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 18e^{2t} \\ 30e^{2t} \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

$$e^{\mathbf{A}t} = \frac{1}{4}\begin{bmatrix} -e^{-t} + 5e^{3t} & e^{-t} - e^{3t} \\ -5e^{-t} + 5e^{3t} & 5e^{-t} - e^{3t} \end{bmatrix}$$

22. Repeat Problem 21, but with $\mathbf{f}(t)$ replaced with $\begin{bmatrix} 28e^{-t} \\ 20e^{3t} \end{bmatrix}$.

23. $\mathbf{A} = \begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 7 \\ 5 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix},$

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 + 3t & -t \\ 9t & 1 - 3t \end{bmatrix}$$

24. Repeat Problem 23, but with $\mathbf{f}(t) = \begin{bmatrix} 0 \\ t^{-2} \end{bmatrix}$ and $\mathbf{x}(1) = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

25. $\mathbf{A} = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 4t \\ 1 \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos t + 2\sin t & -5\sin t \\ \sin t & \cos t - 2\sin t \end{bmatrix}$$

26. Repeat Problem 25, but with $\mathbf{f}(t) = \begin{bmatrix} 4\cos t \\ 6\sin t \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

27. $\mathbf{A} = \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix},\ \mathbf{f}(t) = \begin{bmatrix} 36t^2 \\ 6t \end{bmatrix},\ \mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 + 2t & -4t \\ t & 1 - 2t \end{bmatrix}$$

28. Repeat Problem 27, but with $\mathbf{f}(t) = \begin{bmatrix} 4\ln t \\ t^{-1} \end{bmatrix}$ and $\mathbf{x}(1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

29. $\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} \sec t \\ 0 \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$

30. $\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} t\cos 2t \\ t\sin 2t \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix}$$

31. $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 6e^t \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 2te^t & (3t+2t^2)e^t \\ 0 & e^t & 2te^t \\ 0 & 0 & e^t \end{bmatrix}$$

32. $\mathbf{A} = \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 0 \\ 2e^{2t} \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} e^t & 3te^t & (-13-9t)e^t + 13e^{2t} \\ 0 & e^t & -3e^t + 3e^{2t} \\ 0 & 0 & e^{2t} \end{bmatrix}$$

33. $\mathbf{A} = \begin{bmatrix} 0 & 4 & 8 & 0 \\ 0 & 0 & 3 & 8 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $\mathbf{f}(t) = 30\begin{bmatrix} t \\ t \\ t \\ t \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 4t & 8t+6t^2 & 32t^2+8t^3 \\ 0 & 1 & 3t & 8t+6t^2 \\ 0 & 0 & 1 & 4t \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

34. $\mathbf{A} = \begin{bmatrix} 0 & 4 & 8 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$, $\mathbf{f}(t) = \begin{bmatrix} 0 \\ 6t \\ 0 \\ e^{2t} \end{bmatrix}$, $\mathbf{x}(0) = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \end{bmatrix}$,

$$e^{\mathbf{A}t} = \begin{bmatrix} 1 & 4t & 4(-1+e^{2t}) & 16t(-1+e^{2t}) \\ 0 & 1 & 0 & 4(-1+e^{2t}) \\ 0 & 0 & e^{2t} & 4te^{2t} \\ 0 & 0 & 0 & e^{2t} \end{bmatrix}$$

5.6 Application Automated Variation of Parameters

The application of the variation of parameters formula in Eq. (28) encourages s mechanical an approach as to encourage especially the use of a computer algebı system. The following *Mathematica* commands were used to check the results i Example 4 of this section.

```
A = {{4,2}, {3,-1}};
x0 ={{7}, {3}};
f[t_] := {{-15 t Exp[-2t]},{-4 t Exp[-2t]}};
exp[A_] := MatrixExp[A]
x = exp[A*t].(x0 + Integrate[exp[-A*s].f[s], {s,0,t}])
```

The matrix exponential commands illustrated in the Section 5.5 application provid the basis for analogous *Maple* and MATLAB computations. You can then chec routinely the answers for Problems 17 through 34 of this section.