Solutions Homework 9

Sec. 9.5 (3) $u_t = 2u_{xx}$, $0<x<1$, $t>0$, $u(0,t) = u(1,t) = 0$

$u(x,0) = 5\sin\pi x - \frac{1}{5}\sin 3\pi x.$

$f(x) = 5\sin\pi x - \frac{1}{5}\sin 3\pi x$ alread given as a sine serie

with $b_1 = 5$ and $b_3 = -\frac{1}{5}$, $b_n = 0$ otherwise.

$$\Rightarrow \boxed{u(x,t) = 5e^{-\pi^2 2t}\sin\pi x - \frac{1}{5}e^{-9\pi^2\cdot 2t}\sin 3\pi x}$$

(10) $5u_t = u_{xx}$ $\quad 0<x<10$

$u(0,t) = u(10,t) = 0$ $\qquad k = \frac{1}{5}$

$u(x,0) = f(x) = 4x.$

$$b_n = \frac{2}{10}\int_0^{10} f(x)\sin\frac{n\pi}{10}x\,dx = \frac{1}{5}\int_0^{10} 4x\sin\frac{n\pi}{10}x\,dx = \frac{4}{5}\left[-x\cdot\frac{10}{\pi n}\cos\frac{n\pi}{10}x\Big|_0^{10}\right.$$

$$\left.+\frac{10}{n\pi}\int_0^{10}\cos\frac{n\pi}{10}x\,dx\right] = \frac{4}{5}\cdot\frac{(-100)}{\pi^2 n}\cos n\pi + \cancel{\frac{100}{n^2\pi^2}\sin\frac{n\pi}{10}x\Big|_0^{10}}$$

$$= \frac{8(-1)^n}{\pi n}$$

$$\boxed{u(x,t) = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\frac{n^2\pi^2}{100}\cdot\frac{1}{5}t}\sin\frac{n\pi}{10}x}$$

(13) $L = 40$

$u_t = k u_{xx}$

$u(0,t) = 0$

$u(40,t) = 0$

$u(x,0) = 100$

a. $b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx = \frac{1}{20}\int_0^{40} 100\sin\frac{n\pi}{40}x\,dx = -5\cdot\frac{40}{n\pi}\cos\frac{n\pi x}{40}\Big|_0^{40}$

$= -\frac{200}{\pi n}(\cos n\pi - 1) = \begin{cases} 0 & n \text{ even} \\ \frac{400}{\pi n} & n \text{ odd.} \end{cases}$

$$\boxed{u(x,t) = \frac{400}{\pi}\sum_{n\text{ odd}} \frac{1}{n} e^{-\frac{n^2\pi^2}{1600}kt}\sin\frac{n\pi}{40}x}$$

b. Copper $k = 1.15\ \text{cm}^2/\text{s}$. After 5 min. = 300 s, the temp. at midpoint is

$$u(20,300) = \frac{400}{\pi}\sum_{n\text{ odd}}\frac{1}{n}e^{-\frac{n^2\pi^2 1.15\cdot 300}{1600}}\sin\frac{n\pi}{2}$$

this will be very close to the 1st term:

$$\frac{400}{\pi}e^{-\frac{\pi^2 1.15\cdot 300}{1600}} \simeq 14.08$$

c. Concrete: $k = 0.005$, using first term

$$u(20,t) = 15 \approx \frac{400}{\pi}e^{-\frac{\pi^2\cdot(0.005)t}{1600}} = 15$$

$$e^{\frac{-\pi^2(0.005)t}{1600}} = \frac{15\pi}{400}$$

$$\frac{-\pi^2(0.005)t}{1600} = \ln\left(\frac{15\pi}{400}\right)$$

$$t = -\frac{1600}{0.005\,\pi^2}\ln\left(\frac{15\pi}{400}\right)$$

$= 69342$ seconds

≈ 19.26 hrs. ?

(14) $L = 50$, $k = 1.15$, insulated ends.

$u(x,0) = f(x) = 2x$

a. $$a_0 = \frac{1}{50}\int_0^{50} 2x\,dx = \frac{x^2}{50}\Big|_0^{50} = 50$$

$$a_n = \frac{2}{50}\int_0^{50} 2x\cos\frac{n\pi}{50}x\,dx = 12.5\left[\cancel{x\frac{50}{n\pi}\sin\frac{n\pi}{50}x}\Big|_0^{50} - \frac{50}{n\pi}\int_0^{50}\sin\frac{n\pi}{50}x\,dx\right]$$

$$= 12.5\left(\frac{50}{n\pi}\right)^2\cos\frac{n\pi}{50}x\Big|_0^{50} = \frac{31{,}250}{\pi^2 n^2}(\cos n\pi - 1) = \begin{cases} 0 & n \text{ even} \\ -\dfrac{62{,}500}{\pi^2 n^2} & n \text{ odd}\end{cases}$$

$$\boxed{u(x,t) = 50 - \frac{62{,}500}{\pi^2}\sum_{n\,\text{odd}}\frac{1}{n^2}e^{-\frac{n^2\pi^2(1.15)}{2{,}500}t}\sin\frac{n\pi}{50}x}$$

b. at $x = 10$, after $1 \text{ min} = 60 s$

$$\boxed{u(10,60) = 50 - \frac{62{,}500}{\pi^2} \sum_{n \text{ odd}} \frac{1}{n^2} e^{-\frac{n^2\pi^2(1.15)60}{2{,}500}} \sin\frac{n\pi}{5}}$$

c. to do this, use first term to approximate $u(x,t)$ and set

$\sin\frac{\pi}{5} = 0.588$

$$50 - \frac{62{,}500}{\pi^2} \cdot e^{-\frac{\pi^2(1.15)t}{2{,}500}} = 45$$

and solve for t

$\sin\frac{\pi}{5} = 0.588$

$$\frac{62{,}500}{\pi^2} e^{-\frac{\pi^2(1.15)t}{2{,}500}} \cdot \checkmark = 5$$

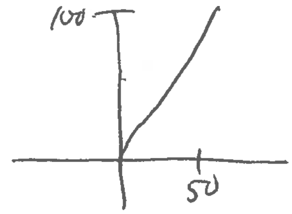
$$e^{-\frac{\pi^2(1.15)t}{2{,}500}} = \frac{5\pi^2}{62{,}500 \cdot (0.588)}$$

$$\boxed{t = -\frac{2{,}500}{\pi^2(1.15)} \ln\left(\frac{5\pi^2}{62{,}500 \cdot (0.588)}\right)}$$

(18) $L=50$, $k=1$, $u(x,0)=0$

$u(0,t)=0$

$u(50,t)=100$



First we find u_{ss}:

$u_{ss} = 2x$

Now we solve
$$v_t = v_{xx}$$
$$v(0,t)=0$$
$$v(50,t)=0$$
$$v(x,0)=f(x)-u_{ss}(x) = -2x:$$

$$b_n = \frac{2}{50}\int_0^{50} -2x \sin\frac{n\pi}{50}x\,dx = \frac{-4}{50}\left[-x\cdot\frac{50}{n\pi}\cos\frac{n\pi}{50}x\Big|_0^{50} + \frac{50}{n\pi}\int_0^{50}\cos\frac{n\pi}{50}x\,dx\right]$$

$$= \frac{-4}{50}\left[-\frac{50^2}{\pi n}\cos n\pi + \cancel{\left(\frac{50}{n\pi}\right)^2 \sin\frac{n\pi}{50}x\Big|_0^{50}}\right]$$

$$= \frac{200\cos n\pi}{\pi n} = (-1)^n\frac{200}{\pi n}.$$

$$\Rightarrow \quad v(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-\frac{n^2\pi^2}{2500}t}\sin\frac{n\pi}{50}x$$

$$u(x,t) = v(x,t) + u_{ss}(x)$$
$$= 2x + \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{\frac{-n^2\pi^2}{2500}t}\sin\frac{n\pi}{50}x$$

or by shifting a negative sign as in book

$$2x - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{\frac{-n^2\pi^2}{2500}t}\sin\frac{n\pi}{50}x$$

(24a) $u_t = k u_{xx}$ $\quad 0 \le x \le L$

$u(0,t) = 0$

$u_x(L,t) = 0$ ← insulated at $x = L$

$u(x,0) = f(x)$

a. Look for solutions of form $v(x,t) = X(x)T(t)$

with boundary conditions: $\left.\begin{array}{l} v(0,t) = X(0)T(t) = 0 \\ v_x(L,t) = X'(L)T(t) = 0 \end{array}\right\}$ all t

$\Rightarrow$ $X(0) = 0$

$X'(L) = 0$.

Plugging XT into the heat eq we will get as before:

$$\frac{T'}{kT} = \frac{X''}{X} = \text{constant} = -\lambda$$

$$\Rightarrow \quad X'' + \lambda X = 0$$

$$T' + \lambda k T = 0$$

Boundary Conditions: $X(0) = 0$, $X'(0) = 0$

$X'' + \lambda X = 0$

$X(0) = 0$

$X'(0) = 0$

If $\lambda = 0$ $\quad X = c_1 x + c_2$.

$X(0) = c_2 = 0$

$X'(L) =$ $c_1 L = 0 \Rightarrow c_1 = 0$

no nontrivial soln's.

If $\lambda < 0$

$$X = c_1 e^{\sqrt{\lambda} x} + c_2 e^{-\sqrt{\lambda} x}$$

$$X' = c_1 \sqrt{\lambda} e^{\sqrt{\lambda} x} - c_2 \sqrt{\lambda} e^{-\sqrt{\lambda} x}$$

$$X(0) = c_1 + c_2 = 0$$

$$X'(L) = c_1 \sqrt{\lambda} e^{\sqrt{\lambda} L} - c_2 \sqrt{\lambda} e^{-\sqrt{\lambda} L} = 0$$

$$\begin{pmatrix} 1 & 1 \\ \sqrt{\lambda} e^{\sqrt{\lambda} L} & -\sqrt{\lambda} e^{-\sqrt{\lambda} L} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

But the determinant of matrix is $-\sqrt{\lambda} e^{-\sqrt{\lambda} L} - \sqrt{\lambda} e^{\sqrt{\lambda} L}$

$$= -\sqrt{\lambda}\left(e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}}\right) \neq 0 \quad \Rightarrow \quad c_1 = c_2 = 0.$$

If $\lambda > 0$

$$X = c_1 \cos\sqrt{\lambda} x + c_2 \sin\sqrt{\lambda} x$$

$$X' = -\sqrt{\lambda} c_1 \sin\sqrt{\lambda} x + \sqrt{\lambda} c_2 \cos\sqrt{\lambda} x$$

$$X(0) = c_1 = 0$$

$$\Rightarrow X = c_2 \sin\sqrt{\lambda} x$$

$$\cancel{X(L) = c_2 \sin\sqrt{\lambda} L = 0}$$

$$X'(0) = \sqrt{\lambda} c_2 \cos\sqrt{\lambda} L = 0$$

$$\Rightarrow \cos\sqrt{\lambda} L = 0$$

$$\sqrt{\lambda} L = \frac{n\pi}{2} \quad n \text{ odd.}$$

$$\Rightarrow \sqrt{\lambda} = \frac{n\pi}{2L} \quad n \text{ odd}$$

$$\Rightarrow X = \sin \frac{n\pi}{2L} x \quad n \text{ odd}$$

Sec 9.6

(6) $y_{tt} = 100\, y_{xx} \quad 0<x<\pi, \quad t>0$

$y(0,t) = y(\pi,t) = 0$

$y(x,0) = x(\pi - x)$

$y_t(x,0) = 0$

$a^2 = 100, \quad a = 10, \quad L = \pi.$

$f(x) = x(\pi - x).$

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi x(\pi - x)\sin nx\,dx$$

$$= \frac{2\cancel{\pi}}{\cancel{\pi}}\int_0^\pi x\sin nx\,dx - \frac{2}{\pi}\int_0^\pi x^2\sin nx\,dx$$

$$= 2\left[-\frac{x}{n}\cos nx\Big|_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right] - \frac{2}{\pi}\left[-\frac{x^2}{n}\cos nx\Big|_0^\pi + \frac{2}{n}\int_0^\pi x\cos nx\,dx\right]$$

$$= 2\left[-\frac{\pi}{n}\cos n\pi + \cancel{\frac{1}{n^2}\sin nx\Big|_0^\pi}\right] - \frac{2}{\pi}\left[-\frac{\pi^2}{n}\cos n\pi + \frac{2}{n}\left(\cancel{\frac{x}{n}\sin nx\Big|_0^\pi} - \frac{1}{n}\int_0^\pi \sin nx\,dx\right)\right]$$

$$= \cancel{-\frac{2\pi}{n}(-1)^n} - \frac{2}{\pi}\left[\cancel{-\frac{\pi^2}{n}(-1)^n} + \frac{2}{n}\left(+\frac{1}{n^2}\cos nx\Big|_0^\pi\right)\right]$$

$$= -\frac{4}{\pi n^3}(\cos n\pi - 1) = \begin{cases} 0 & n \text{ even} \\ \frac{8}{\pi n^3} & n \text{ odd} \end{cases}$$

$$u(x,t) = \frac{8}{\pi}\sum_{n\text{ odd}} \frac{1}{n^3}\cos\left(\frac{n\pi 10}{\cancel{\pi}}t\right)\sin nx$$

or $\left| \frac{8}{\pi}\sum_{n\text{ odd}} \frac{1}{n^3}\cos(10nt)\sin nx \right|$

(13) F twice diff'l all x in $\mathbb{R}$.

$y(x,t) = F(x+at)$, $\quad z(x,t) = F(x-at)$ call this one $z(x,t)$

$y_t = F'(x+at)\cdot a$

$y_{tt} = F''(x+at)\cdot a^2$

$y_x = F'(x+at)$

$y_{xx} = F''(x+at)$

~~So $y_{tt} + a^2 y_{xx} = F''(x+at)\cdot a^2 + a^2$~~

So $y_{tt} = a^2 F''(x+at)$

and $a^2 y_{xx} = a^2 F''(x+at)$.

$z_t = F'(x-at)(-a)$

$z_{tt} = F''(x-at)(-a)(-a) = F''(x-at)a^2$

$z_x = F'(x-at)$

$z_{xx} = F''(x-at)$

$\Rightarrow z_{tt} = a^2 z_{xx}$.

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(14) F $2L$-periodic and odd

$y(x,t) = \frac{1}{2}[F(x+at) + F(x-at)]$

$y(0,t) = \frac{1}{2}[F(at) + F(-at)] = \frac{1}{2}[F(at) - F(at)] = 0$ (since F odd)

$y(L,t) = \frac{1}{2}[F(L+at) + F(L-at)] = \frac{1}{2}[F(L+at) - F(-L+at)]$ (F odd)

$= \frac{1}{2}[F(L+at) - F(-L+at+2L)]$ (F 2L-period)

$= \frac{1}{2}[F(L+at) - F(L+at)] = 0$.

$$y(x,0) = \frac{1}{2}\left[F(x) + F(x)\right] = F(x)$$

$$y_t(x,t) = \frac{1}{2}\left[F'(x+at)\,a + F'(x-at)(-a)\right]$$

$$y_t(x,0) = \frac{1}{2}\left[F'(x)\,a - F'(x)\,a\right] = 0$$