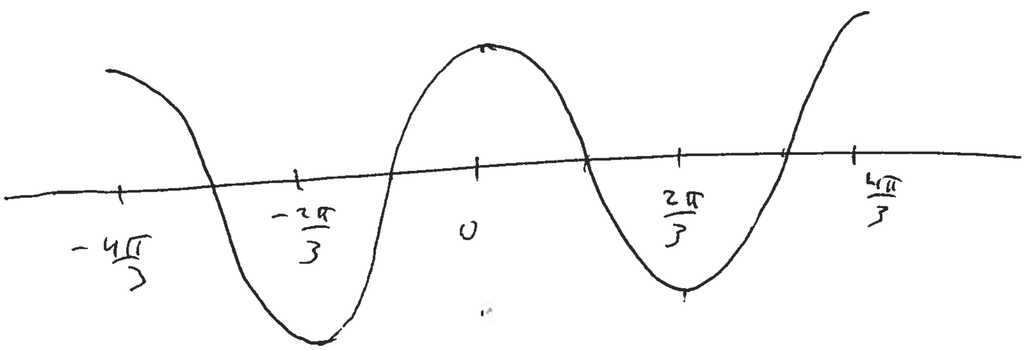
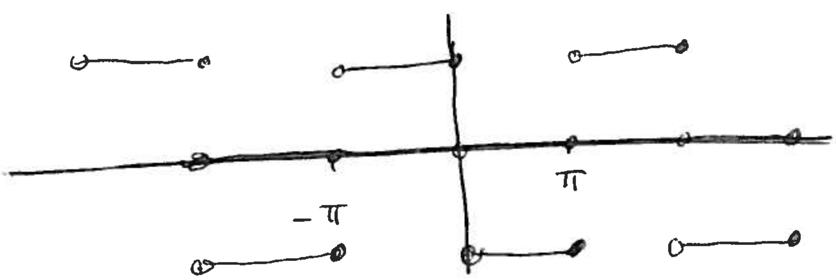
Homework 8 Solutions

Sec. 9.1 (3) $f(t) = \cos\frac{3}{2}t$. f is periodic, the period is T where $\frac{3}{2}T = 2\pi$ or $\boxed{T = \frac{4\pi}{3}}$



(12) $f(t) = \begin{cases} 3 & -\pi < t \le 0 \\ -3 & 0 < t \le \pi \end{cases}$



f is odd so $a_n = 0$ all n.

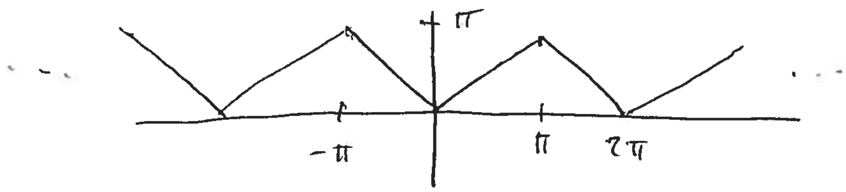
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt \quad (\text{since } f(t)\sin nt \text{ is even})$$

$$= \frac{2}{\pi}\int_0^{\pi} -3\sin nt\,dt = \frac{+6}{\pi n}\cos nt\Big|_0^{\pi} = \frac{6}{\pi n}(\cos n\pi - 1)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{-12}{\pi n} & n \text{ odd.} \end{cases}$$

$$\boxed{\frac{-12}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin nt}$$

(17) $f(t) = |t| \qquad -\pi \le t \le \pi$



f is even so $b_n = 0$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(t)\,dt = \frac{1}{\pi}\int_0^\pi t\,dt \qquad (\text{since } |t| = t \text{ on } [0,\pi])$$

$$= \frac{t^2}{2\pi}\Big|_0^\pi = \frac{\pi}{2}$$

$u = t \qquad du = dt$

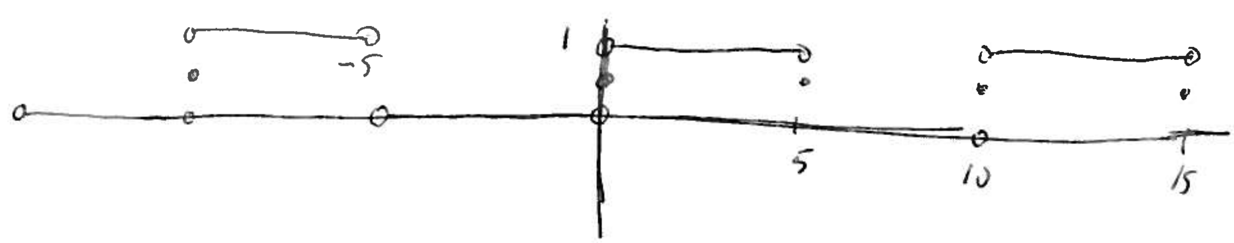
$dv = \cos nt \qquad v = \frac{\sin nt}{n}$

$$a_n = \frac{2}{\pi}\int_0^\pi t\cos nt = \frac{2}{\pi}\left[\cancel{t\,\frac{\sin nt}{n}}\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nt\,dt\right]$$

$$= -\frac{2}{\pi n}\cdot\frac{-\cos nt}{n}\Big|_0^\pi = \frac{2}{\pi n^2}(\cos n\pi - 1) = \begin{cases} 0 & n \text{ even} \\ -\frac{4}{\pi n^2} & n \text{ odd}\end{cases}$$

$$\boxed{\frac{\pi}{2} - \frac{4}{\pi}\sum_{n\ \text{odd}} \frac{1}{n^2}\cos nt}$$

Sec 9.2 (2) $f(t) = \begin{cases} 0 & -5 < t < 0 \\ 1 & 0 < t < 5 \end{cases}$



$$a_0 = \frac{1}{10}\int_{-5}^{5} f(t)\,dt = \frac{1}{10}\int_{0}^{5} dt = \frac{1}{2}$$

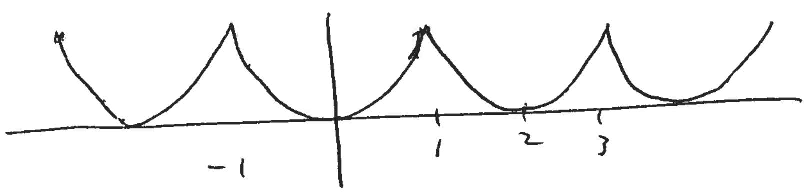
$$a_n = \frac{1}{5}\int_{-5}^{5} f(t)\cos\frac{n\pi}{5}t\,dt = \frac{1}{5}\int_{0}^{5} \cos\frac{n\pi}{5}t\,dt = \frac{1}{5}\,\frac{5}{n\pi}\sin n\pi t\Big|_0^5 = 0$$

$$b_n = \frac{1}{5}\int_{-5}^{5} f(t)\sin\frac{n\pi}{5}t\,dt = \frac{1}{5}\int_{0}^{5} \sin\frac{n\pi}{5}\,dt = -\frac{1}{5}\cdot\frac{5}{n\pi}\cos\frac{n\pi}{5}t\Big|_0^5$$

$$= -\frac{1}{\pi n}\left(\cos n\pi - 1\right) = \begin{cases} 0 & n \text{ even} \\ \dfrac{2}{\pi n} & n \text{ odd.} \end{cases}$$

$$\boxed{\frac{1}{2} + \frac{2}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin\frac{n\pi}{5}t}$$

(9) $f(t) = t^2 \qquad -1 < t < 1$



f is even so $b_n = 0$.

$$u = t \quad dv = \sin n\pi t\,dt \quad v = -\frac{\cos n\pi t}{n\pi}$$

$$a_0 = \frac{1}{1}\int_0^1 t^2\,dt = \frac{t^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$a_n = 2\int_0^1 t^2\cos n\pi t\,dt = 2\left[\cancel{\frac{t^2\sin n\pi t}{n\pi}\Big|_0^1} - \frac{2}{n\pi}\int_0^1 t\sin n\pi t\,dt\right]$$

$$u = t^2 \quad dv = \cos n\pi t\,dt \quad du = 2t\,dt \quad v = \frac{\sin n\pi t}{n\pi}$$

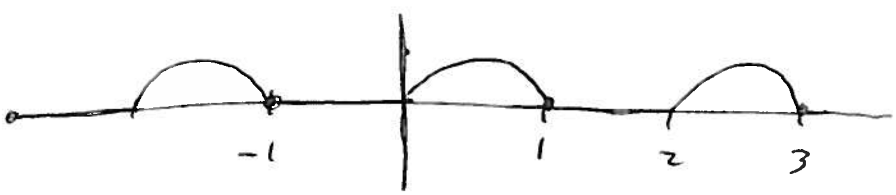
$$= -\frac{4}{n\pi}\left[-\frac{t}{n\pi}\cos n\pi t\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos n\pi t\, dt\right]$$

$$= \frac{-4}{n\pi}\left[\frac{-1}{n\pi}\cos n\pi + \cancel{\frac{1}{n\pi}\frac{\sin n\pi t}{n\pi}}\Big|_0^1\right]$$

$$= \frac{4}{\pi^2 n^2}\cos n\pi = \frac{4(-1)^n}{\pi^2 n^2}$$

$$\boxed{\underset{(t^2)}{f(t)} = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi t}$$

(13) $f(t) = \begin{cases} 0 & -1 < t < 0 \\ \sin \pi t & 0 < t < 1 \end{cases}$



$$a_0 = \frac{1}{2}\int_{-1}^{1} f(t)\, dt = \frac{1}{2}\int_0^1 \sin \pi t\, dt = -\frac{1}{2}\frac{\cos \pi t}{\pi}\Big|_0^1 = \frac{-1}{2\pi}(-1-1) = \frac{1}{\pi}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos n\pi t\, dt = \int_0^1 \sin \pi t \cos n\pi t\, dt$$

If $n=1$: $\frac{1}{2}\int_0^1 \sin 2\pi t\, dt = -\frac{1}{4\pi}\cos 2\pi t\Big|_0^1 = 0$

If $n>1$, using #31, integral table: $a_n = -\frac{\cos(\pi - n\pi)t}{2(\pi - n\pi)} - \frac{\cos(\pi + n\pi)t}{2(\pi + n\pi)}\Big|_0^1$

$$= \frac{-\cos((n-1)\pi)+1}{2\pi(n-1)} - \frac{\cos((n+1)\pi)-1}{2\pi(n+1)}$$, If n odd, $n-1$, $n+1$ even terms are zero

n even:

$$a_n = \frac{2}{\cancel{2}\pi(n-1)} + \frac{\cancel{2}}{\cancel{2}\pi(n+1)} = \frac{1}{\pi}\left(\frac{1}{n-1} + \frac{1}{n+1}\right) = \frac{2}{\pi}\cdot\frac{n}{n^2-1}$$

$$b_n = \int_{-1}^{1} f(t)\sin n\pi t\,dt = \int_0^1 \sin \pi t \sin n\pi t\,dt$$

$$= 0 \quad \text{if} \quad n \neq 1$$

If $n = 1$: $\quad b_1 = \int_0^1 \sin^2 \pi t\,dt = \frac{1}{2}$

$$\boxed{f(t) = \frac{1}{\pi} + \frac{2}{\pi}\sum_{n \text{ even}} \frac{n}{n^2-1}\cos n\pi t + \frac{1}{2}\sin \pi t}$$

25a $\quad f(t)$ – 2π periodic, $\quad f(t) = t^3, \quad -\pi < t < \pi$

f odd, so $a_n = 0$ all n.

$$b_n = \frac{2}{\pi}\int_0^{\pi} t^3 \sin nt\,dt = \frac{2}{\pi}\left[-\frac{t^3}{n}\cos nt \Big|_0^{\pi} + \frac{3}{n}\int_0^{\pi} t^2 \cos nt\,dt\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi^3}{n}\cos n\pi + \frac{3}{n}\left(\cancel{\frac{t^2}{n}\sin nt \Big|_0^{\pi}} - \frac{2}{n}\int_0^{\pi} t \sin nt\,dt\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi^3}{n}(-1)^n - \frac{6}{n^2}\left(-\frac{t}{n}\cos nt\Big|_0^{\pi} + \cancel{\frac{1}{n}\int_0^{\pi}\cos nt\,dt}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^3}{n}(-1)^{n+1} + \frac{6}{n^3}\pi(-1)^n\right] = \frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n$$

$$\boxed{\sum_{n=1}^{\infty}\left(\frac{2\pi^2}{n}(-1)^{n+1} + \frac{12}{n^3}(-1)^n\right)\sin nt}$$

Sec. 9.3 (1) $f(t) = 1, \quad 0 < t < \pi$

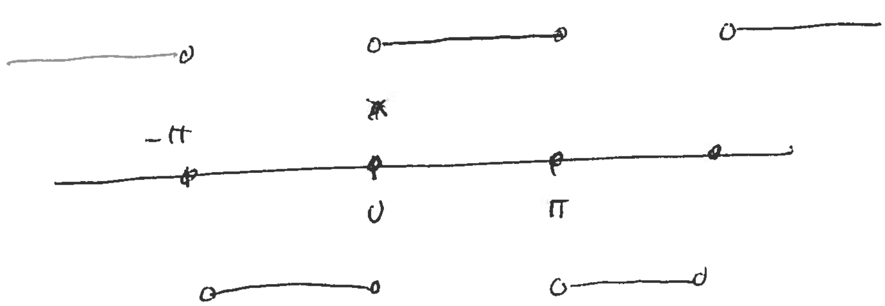
Cosine series:

$$a_0 = \frac{1}{\pi}\int_0^{\pi} dt = 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \cos nt \, dt = \frac{2}{\pi} \frac{\sin nt}{n}\Big|_0^{\pi} = 0$$

Cosine Series: $\boxed{f(t) = 1}$

Sine series:



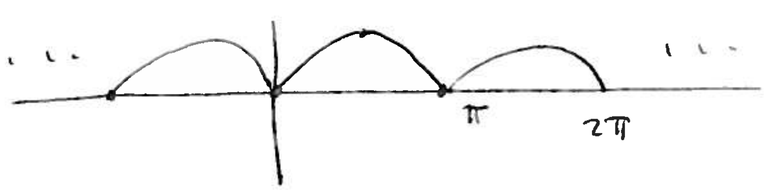
$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin nt = \frac{2}{\pi} \cdot -\frac{\cos nt}{n}\Big|_0^{\pi} = -\frac{2}{\pi n}(\cos n\pi - 1)$$

$$= \begin{cases} 0 & n \text{ even} \\ \frac{4}{\pi n} & n \text{ odd.} \end{cases}$$

$$\boxed{f(t): \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\sin nt}$$

(7) $f(t) = t(\pi - t) \quad 0 < t < \pi$

Cosine Series



$$a_0 = \frac{1}{\pi}\int_0^\pi t(\pi-t)\,dt = \frac{1}{\pi}\int_0^\pi \pi t - t^2\,dt = \frac{1}{\pi}\left(\frac{\pi t^2}{2} - \frac{t^3}{3}\right)\Big|_0^\pi$$

$$= \frac{1}{\pi}\left(\frac{\pi^3}{2} - \frac{\pi^3}{3}\right) = \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi}\int_0^\pi t(\pi-t)\cos nt\,dt = 2\int_0^\pi t\cos nt\,dt - \frac{2}{\pi}\int_0^\pi t^2\cos nt\,dt$$

$$= 2\left[\cancel{\frac{t}{n}\sin nt}\Big|_0^\pi - \frac{1}{n}\int_0^\pi \sin nt\,dt\right] - \frac{2}{\pi}\left[\cancel{\frac{t^2}{n}\sin nt}\Big|_0^\pi - \frac{2}{n}\int_0^\pi t\sin nt\,dt\right]$$

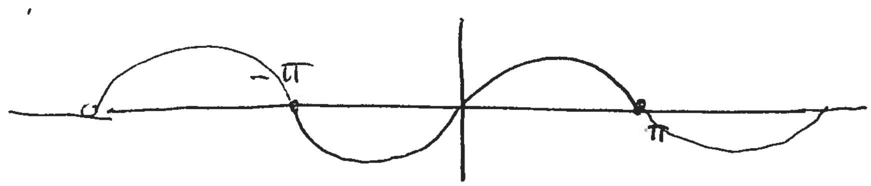
$$= -\frac{2}{n}\left(-\frac{\cos nt}{n}\Big|_0^\pi\right) + \frac{4}{\pi n}\left[-\frac{t}{n}\cos nt\Big|_0^\pi + \cancel{\frac{1}{n}\int_0^\pi \cos nt\,dt}\right]$$

$$= \frac{2}{n^2}(\cos n\pi - 1) - \frac{4\pi}{\pi n^2}\cos n\pi = \frac{2}{n^2}\left((-1)^n - 1 - 2(-1)^n\right) = \frac{2}{n^2}\left(-1-(-1)^n\right)$$

$$= \begin{cases} 0 & n \text{ odd} \\ -\frac{4}{n^2} & n \text{ even} \end{cases}$$

$$\boxed{f(t) = \frac{\pi^2}{6} - 4\sum_{n \text{ even}} \frac{1}{n^2}\cos nt}$$

Sine Series:



$$b_n = \frac{2}{\pi}\int_0^\pi t(\pi-t)\sin nt\,dt = 2\int_0^\pi t\sin nt\,dt - \frac{2}{\pi}\int_0^\pi t^2\sin nt$$

$$= 2\left(-\frac{t}{n}\cos nt\Big|_0^{\pi} + \frac{1}{n}\cancel{\int_0^{\pi}\cos nt\,dt}\right) - \frac{2}{\pi}\left[-\frac{t^2}{n}\cos nt\Big|_0^{\pi} + \frac{2}{n}\int_0^{\pi} t\cos nt\,dt\right]$$

$$= -\frac{2\pi}{n}(-1)^n - \frac{2}{\pi}\left[-\frac{\pi^2}{n}(-1)^n + \frac{2}{n}\left(\cancel{\frac{t}{n}\sin nt\Big|_0^{\pi}} - \frac{1}{n}\int_0^{\pi}\sin nt\,dt\right)\right]$$

$$= \cancel{-\frac{2\pi}{n}(-1)^n} \cancel{+\frac{2\pi}{n}(-1)^n} + \frac{-4}{\pi n^2}\,\frac{\cos nt}{n}\Big|_0^{\pi} = \frac{-4}{\pi n^3}\left((-1)^n - 1\right)$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{8}{\pi n^3} & n \text{ odd.} \end{cases}$$

$$\boxed{f(t) = \frac{8}{\pi}\sum_{n\ \text{odd}} \frac{1}{n^3}\sin nt}$$

(19) $$t = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\sin nt \qquad -\pi < t < \pi$$

$$\int_0^t t\,dt = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_0^t \sin nt\,dt$$

$$\frac{t^2}{2} = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(-1)\cos nt}{n^2}\Big|_0^t = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nt - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{t^3}{6} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\int_0^t \cos nt\,dt - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,t$$

$$\frac{t^3}{6} = 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nt - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,t$$

One more time:

$$\frac{t^4}{24} = 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}\int_0^t \sin nt\,dt - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,t^2$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^n(-1)(\cos nt - 1)}{n^4} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,t^2$$

$$= -2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos nt + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\,t^2$$

Now, from problem 9, sec. 9.2 (also done in class)

$$t^2 = \frac{1}{4} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi t \qquad -1 < t < 1$$

Substituting $t = 0$ gives

$$0 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} \Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

substituting above gives.

$$\frac{t^4}{24} = \frac{\pi^2 t^2}{12} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}\cos nt + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

20 Substituting $t = \pi$ gives

$$\frac{\pi^4}{24} = \frac{\pi^4}{12} - 2\sum_{n=1}^{\infty}\frac{\cancel{(-1)^n(-1)^n}}{n^4} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4}$$

$$2\left(\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^4}\right) = \frac{\pi^4}{12} - \frac{\pi^4}{24} = \frac{\pi}{24}$$

$$2\sum_{n\ \text{odd}} \frac{2}{n^4} = \frac{\pi^4}{24}$$

$$\boxed{\sum_{n\ \text{odd}} \frac{1}{n^4} = \frac{\pi^4}{96}}$$

Substituting $x = \pi/2$:

$$\frac{\pi^4}{24\cdot 16} = \frac{\pi^4}{48} - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4} \overbrace{\cos\frac{n\pi}{2}}^{\overset{n\ \text{odd}}{0\ =}} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

$$2\sum_{\cancel{n\ \text{odd}}\ \text{even}}^{\infty} \frac{(-1)^n}{n^4}\cos\frac{n\pi}{2} - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = \frac{\pi^4}{48} - \frac{\pi^4}{384} = \frac{7\pi^4}{384}$$

$$\sum_{n\ \text{even}} \frac{\cos n\pi/2}{n^4} - \sum_{n=1}^{\infty}\frac{(-1)^n}{n^4} = \frac{7\pi^4}{768}$$

$$\sum_{n\ \text{even}} \frac{\cos n\pi/2}{n^4} - \sum_{n\ \text{even}}\frac{1}{n^4} + \underbrace{\sum_{n\ \text{odd}}\frac{1}{n^4}}_{\pi^4/96} = \frac{7\pi^4}{768}$$

Note that in the middle term on the right, the odd terms have a factor of $\cos\frac{n\pi}{2} = 0$, so we only get the even ones. Therefore substitute $m = \frac{n}{2}$, that is $n = 2m$, $m = 1, 2, ..$

and we get

$$\frac{\pi^4}{384} = \frac{\pi^4}{48} - 2\sum_{m=1}^{\infty} \frac{\cancel{(-1)^{2m}}}{(2m)^4} \cos m\pi + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

$$\frac{\pi^4}{384} = \frac{\pi^4}{48} - 2\sum_{m=1}^{\infty} \frac{(-1)^m}{16\, m^4} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$$

$$\frac{\pi^4}{384} = \frac{\pi}{48} + 2\,\frac{1}{16}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m^4} - 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

Let's call $S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$. Then

$$\frac{\pi^4}{384} = \frac{\pi}{48} + 2\left(\frac{1}{16} S - S\right)$$

$$\frac{\pi^4}{384} = \frac{\pi}{48} - \frac{30}{16} S \Rightarrow \frac{15}{8} S = \frac{\pi}{48} - \frac{\pi}{384}$$

$$S = \frac{8}{15}\pi^4\left(\frac{1}{48} - \frac{1}{384}\right) \Rightarrow \boxed{S = \pi^4 \cdot \frac{7}{720}}$$

To get $\sum_{n=1}^{\infty} \frac{1}{n^4}$ Note that

$$\sum_{n \text{ even}} \frac{1}{n^4} = \sum_{n \text{ odd}} \frac{1}{n^4} - S \Rightarrow \sum_{n \text{ even}} \frac{1}{n^4} = \frac{\pi^4}{96} - \frac{7}{720}\pi^4$$

therefore $\sum_{n=1}^{\infty} \frac{1}{n^4} = \sum_{n \text{ odd}} \frac{1}{n^4} + \sum_{n \text{ even}} \frac{1}{n^4}$

$$= \frac{\pi^4}{96} + \frac{\pi^4}{96} - \frac{7}{720}\pi^4$$

$$= \left(\frac{2}{96} - \frac{7}{720}\right)\pi^4$$

$$= \left(\frac{15}{720} - \frac{7}{720}\right)\pi^4$$

$$= \frac{8}{720}\pi^4$$

$$= \frac{\pi^4}{90} \quad !$$