Homework 6 Solutions

Sec. 6.1 (5) $\frac{dx}{dt} = 1-y^2$ $\qquad 1-y^2=0 \to y^2=1,\ y=\pm 1$

$\frac{dy}{dt} = x+2y$ $\qquad x+2y=0 \to y=1,\ x=-2$

$y=-1,\ x=2$

critical points: $(2,-1)$ and $(-2,1)$ Figure 6.1.12

(6) $\frac{dx}{dt} = 2-4x-15y \quad 2-4x-15y=0 \Rightarrow 2-4x=15y$

$\frac{dy}{dt} = 4-x^2 \Rightarrow 4-x^2=0 \qquad x^2=4,\ x=\pm 2$

$y = \frac{2}{15} - \frac{4x}{15}$ so $x=2$ then $y = \frac{2}{15} - \frac{8}{15} = -\frac{2}{5}$

$x=-2 \quad y = \frac{2}{15} + \frac{8}{15} = \frac{2}{3}$

$\Rightarrow$ critical points: $(2,-\frac{2}{5}),\ (-2, 2/3)$ as in Fig. 6.1.18

(17) $\frac{dx}{dt} = y$ $\Rightarrow \vec{x}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\vec{x}$ where $\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\frac{dy}{dt} = -x$

Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2+1=0$, $\lambda = \pm i$

Eigenvector for i: $\left(\begin{array}{cc|c} -i & 1 & 0 \\ -1 & -i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & i & 0 \\ 0 & 0 & 0 \end{array}\right)$, $x_1 = -i x_2$

set $x_2 = i$
$\Rightarrow x_1 = 1$

$\vec{v} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Cx Solution is:

$\vec{x}(t) = e^{it}\vec{v} = (\cos t + i\sin t)\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \left[\cos t\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sin t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

$+ i\left[\sin t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \cos t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$

General Soln:

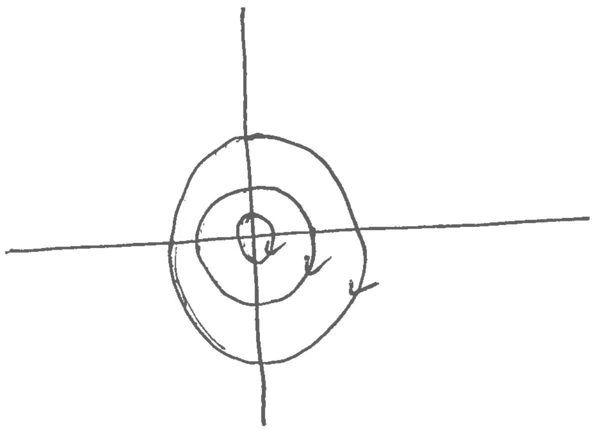
$$\boxed{\vec{x}(t) = c_1\left[\cos t\begin{pmatrix}1\\0\end{pmatrix} - \sin t\begin{pmatrix}0\\1\end{pmatrix}\right] + c_2\left[\sin t\begin{pmatrix}1\\0\end{pmatrix} + \cos t\begin{pmatrix}0\\1\end{pmatrix}\right]}$$

<u>Or</u>

$$\begin{aligned} x_1 &= c_1 \cos t + c_2 \sin t \\ x_2 &= c_2 \cos t - c_1 \sin t \end{aligned}$$

(0, 0) is a stable center

<u>Note</u>: if trajectory starts on x-axis when $t=0$, then
$c_2 = 0$
$x = c_1 \cos t$
$y = -c_1 \sin t$
eq. of circle <u>clockwise</u>



(18) $\dfrac{dx}{dt} = -y$ $\Rightarrow \begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}0 & -1\\ 4 & 0\end{pmatrix}$

$\dfrac{dy}{dt} = 4x$

Eigenvalues: $\begin{vmatrix}-\lambda & -1\\ 4 & -\lambda\end{vmatrix} = \lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

Eigenvector for $2i$: $\left(\begin{array}{cc|c}-2i & -1 & 0\\ 4 & -2i & 0\end{array}\right) \to \left(\begin{array}{cc|c}4 & -2i & 0\\ 0 & 0 & 0\end{array}\right) \to \left(\begin{array}{cc|c}1 & -\frac{1}{2}i & 0\\ 0 & 0 & 0\end{array}\right)$

$x_1 = \frac{1}{2} i x_2$ cx eigenvector $\vec{v} = \begin{pmatrix}i\\2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix} + i\begin{pmatrix}1\\0\end{pmatrix}$

set $x_2 = 2$

complex soln: $\begin{pmatrix}x\\y\end{pmatrix} = (\cos 2t + i\sin 2t)\left[\begin{pmatrix}0\\2\end{pmatrix} + i\begin{pmatrix}1\\0\end{pmatrix}\right] =$

$$= \left[\cos 2t \begin{pmatrix}0\\2\end{pmatrix} - \sin 2t \begin{pmatrix}1\\0\end{pmatrix}\right] + i\left[\sin 2t \begin{pmatrix}0\\2\end{pmatrix} + \cos 2t \begin{pmatrix}1\\0\end{pmatrix}\right]$$

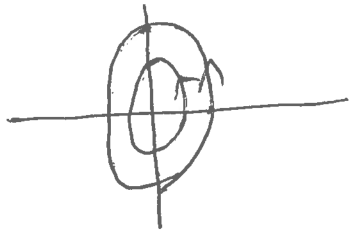
General Solution is

$$\boxed{\begin{pmatrix}x\\y\end{pmatrix} = c_1\left[\cos 2t \begin{pmatrix}0\\2\end{pmatrix} - \sin 2t \begin{pmatrix}1\\0\end{pmatrix}\right] + c_2\left[\sin 2t \begin{pmatrix}0\\2\end{pmatrix} + \cos 2t \begin{pmatrix}1\\0\end{pmatrix}\right]}$$

or
$$\boxed{\begin{aligned} x &= -c_1 \sin 2t + c_2 \cos 2t \\ y &= 2c_1 \cos 2t + 2c_2 \sin 2t \end{aligned}}$$

If we start a trajectory on x axis $(t=0)$, $c_1 = 0$

$$\begin{aligned} x &= c_2 \cos 2t \\ y &= 2c_2 \sin 2t \end{aligned} \Rightarrow x^2 + \frac{y^2}{4} = c_2^2$$ ellipse counterclockwise



Stable center

(20) $$\frac{dx}{dt} = y \qquad \Rightarrow \begin{pmatrix}x\\y\end{pmatrix}' = \begin{pmatrix}0 & 1\\ -5 & -4\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$
$$\frac{dy}{dt} = -5x - 4y$$

eigenvalues: $\begin{vmatrix}-\lambda & 1 \\ -5 & -4-\lambda\end{vmatrix} = \lambda(4+\lambda) + 5 = 0$; $\lambda^2 + 4\lambda + 5 = 0$

$$\lambda = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$$

Eigenvectors for $-1+i$: $\left(\begin{array}{cc|c} 1-i & 1 & 0 \\ -5 & -3-i & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{3}{5}+\frac{1}{5}i & 0 \end{array}\right)$

Eigenvector for $-2+i$:

$$\left(\begin{array}{cc|c} 2-i & 1 & 0 \\ -5 & -2-i & 0\end{array}\right) \to \left(\begin{array}{cc|c} 5 & 2+i & 0 \\ 0 & 0 & 0\end{array}\right) \to \left(\begin{array}{cc|c} 1 & \frac{2}{5}+\frac{1}{5}i & 0 \\ 0 & 0 & 0\end{array}\right)$$

$x_1 = \left(-\frac{2}{5} - \frac{1}{5}i\right) x_2.$

set $x_2 = 5$, $x_1 = -2-i$

$$\vec{v} = \begin{pmatrix} -2-i \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

cx soln. $e^{-2t}(\cos t + i \sin t)\left[\begin{pmatrix} -2 \\ 5 \end{pmatrix} + i \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right]$

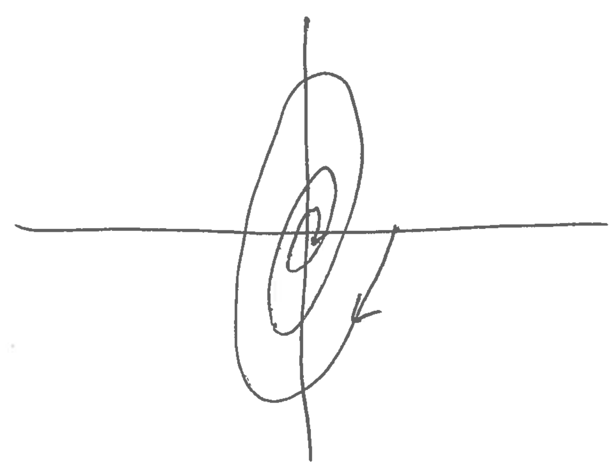
$$= e^{-2t}\left[\cos t \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \sin t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] + i\, e^{-2t}\left[\cos t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} -2 \\ 5 \end{pmatrix}\right]$$

General Soln:

$$\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{-2t}\left[\cos t \begin{pmatrix} -2 \\ 5 \end{pmatrix} - \sin t \begin{pmatrix} -1 \\ 0 \end{pmatrix}\right] + c_2 e^{-2t}\left[\cos t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \sin t \begin{pmatrix} -2 \\ 5 \end{pmatrix}\right]$$

$(0,0)$ is a stable spiral. Plugging $\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ into the vector field we get $\begin{pmatrix} 0 & 1 \\ -5 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -5 \end{pmatrix}$ points down

Clockwise.



(21) Example 5 in my text

$$\frac{dx}{dt} = -ky + x(1-x^2-y^2)$$
$$\frac{dy}{dt} = kx + y(1-x^2-y^2)$$

Suppose $-ky + x(1-x^2-y^2) = 0$

$kx + y(1-x^2-y^2) = 0$

Then $\left[-ky + x(1-x^2-y^2)\right]^2 + \left[kx + y(1-x^2-y^2)\right]^2 = 0$

$$k^2y^2 + x^2(1-x^2-y^2)^2 - \cancel{2kxy(1-x^2-y^2)} + k^2x^2 + y^2(1-x^2-y^2) + \cancel{2kxy(1-x^2-y^2)} = 0$$

$$\Rightarrow k^2y^2 + x^2(1-x^2-y^2)^2 + k^2x^2 + y^2(1-x^2-y^2)^2 = 0$$

Every term is non-negative $\Rightarrow$ every term is zero

$\Rightarrow$ $x = 0$ and $y = 0$.

(22) $\frac{dr}{dt} = r(1-r^2) \Rightarrow \int \frac{dr}{r(1-r^2)} = \int dt = t + C$

$$\Rightarrow \frac{1}{r(1-r^2)} = \frac{1}{r(1-r)(1+r)} = \frac{A}{r} + \frac{B}{1-r} + \frac{C}{1+r}$$

$\Rightarrow A(1-r)(1+r) + Br(1+r) + Cr(1-r) = 1$

$r = 0 \Rightarrow A = 1$ $\quad$ $r = -1 \Rightarrow C = -\frac{1}{2}$

$r = 1 \Rightarrow B = \frac{1}{2}$

$$\int \frac{dr}{r} + \frac{1}{2}\int \frac{dr}{1-r} - \frac{1}{2}\int \frac{dr}{1+r} = t + C$$

$$\ln r - \frac{1}{2}\ln|1-r| - \frac{1}{2}\ln(1+r) = t + c.$$

$$\ln r - \ln(1-r)^{1/2} - \ln(1+r)^{1/2} = t + c \qquad (r<1)$$

$$\ln \frac{r}{(1-r)^{1/2}(1+r)^{1/2}} = t + c$$

$$(*) \quad \frac{r}{\sqrt{1-r^2}} = e^t K \qquad K = e^c.$$

So when $t=0 \Rightarrow \dfrac{r_0}{\sqrt{1-r_0^2}} = K$

From $(*)$ $$\frac{r^2}{1-r^2} = e^{2t}K^2 = e^{2t}\frac{r_0^2}{1-r_0^2}$$

$$\Rightarrow \quad r^2 = (1-r^2)\,\frac{e^{2t}r_0^2}{(1-r_0^3)} = \frac{e^{2t}r_0^2}{1-r_0^2} - r^2\,\frac{e^{2t}r_0^3}{1-r_0^2}$$

$$r^2\left(1 + \frac{e^{2t}r_0^2}{1-r_0^2}\right) = \frac{e^{2t}r_0^2}{1-r_0^2}$$

$$r^2\left(\frac{1-r_0^2+e^{2t}r_0^2}{\cancel{1-r_0^2}}\right) = \frac{e^{2t}r_0^2}{\cancel{1-r_0^2}}$$

$$r^2 = \frac{e^{2t}r_0^2}{1-r_0^2+e^{2t}r_0^2}\cdot\frac{e^{-2t}}{e^{-2t}} = \frac{r_0^2}{r_0^2+(1-r_0^2)e^{-2t}}$$

taking square roots gives (21)

$$r = \frac{r_0}{\sqrt{r_0^2+(1-r_0^2)e^{-2t}}}$$

Sec. 5.3

(1) $\begin{aligned} x_1' &= x_1+2x_2 \\ x_2' &= 2x_1+x_2 \end{aligned} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

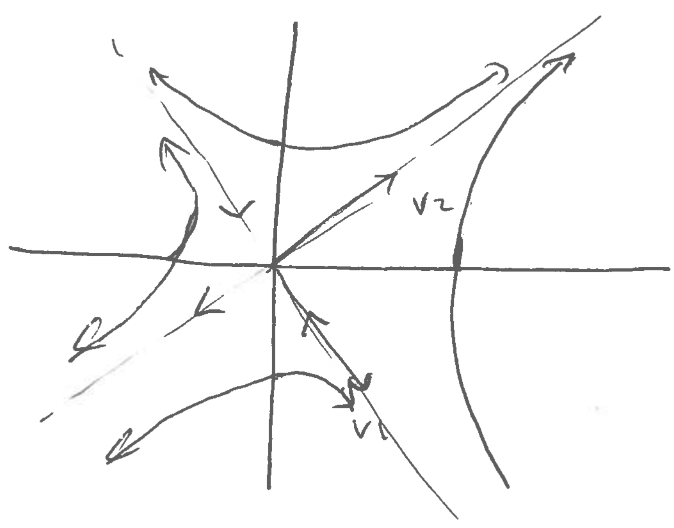
eigenvalues: $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0,\ (1-\lambda)^2 = 4$

$1-\lambda = \pm 2 \Rightarrow \lambda = 1 \pm 2,\ \boxed{\lambda = -1 \text{ and } \lambda = 3}$

Thus O is an unstable saddle.

Eigenvectors: $\lambda = -1$: $\left(\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right) \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},\ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\lambda = 3$ $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ can take $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$



⑧ $x_1' = \cancel{8}\, x_1 - 5x_2$
$x_2' = x_1 - x_2$ $\Rightarrow$ $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = -(1+\lambda)(1-\lambda) + 5 = 0$

$\lambda^2 + 4 = 0$, $\boxed{\lambda = 2i,\ \lambda = -2i}$

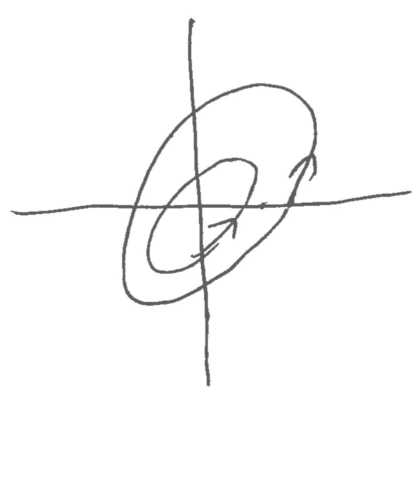
Thus $(0,0)$ stable center.

Eigenvector: $\lambda = 2i$ $\left(\begin{array}{cc|c} 1-2i & -5 & 0 \\ 1 & -1-2i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1-2i & 0 \\ 0 & 0 & 0 \end{array}\right)$

complex eigenvector: $x_1 = (1+2i)x_2$ setting $x_2 = 1$

$\vec{V} = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$

Evaluating the Direction field at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$: $\begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ points upwards so counterclockwise.



(13) $x_1' = 5x_1 - 9x_2$ $\quad\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

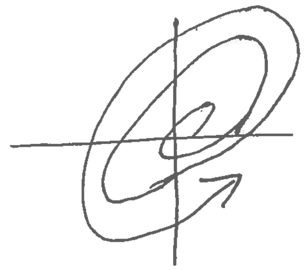
$x_2' = 2x_1 - x_2$

Eigenvalues: $\begin{vmatrix} 5-\lambda & -9 \\ 2 & -1-\lambda \end{vmatrix} = -(\lambda+1)(5-\lambda) + 18 = 0$

$\lambda^2 - 4\lambda + 13 = 0$, $\quad \lambda = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm 6i}{2} = 2 \pm 3i$

eigenvalues are $2+3i$, $2-3i$

$(0,0)$ unstable spiral. $\quad \begin{pmatrix} 5 & -9 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$



(33) (a) A has repeated eigenvalue λ with 2 linearly independent eigenvectors $\vec{v}_1, \vec{v}_2$. Since $\vec{v}_1, \vec{v}_2$ are linearly independent they form a basis for $\mathbb{R}^2$. Thus if $\vec{v}$ is <u>any</u> vector in $\mathbb{R}^2$ $\quad \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2$ $\quad$ some numbers c_1, c_2

Then $A\vec{v} = A(c_1\vec{v}_1 + c_2\vec{v}_2) = A(c_1\vec{v}_1) + A(c_2\vec{v}_2)$

$= c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = c_1 \lambda \vec{v}_1 + c_2 \lambda \vec{v}_2$

$= \lambda(c_1\vec{v}_1 + c_2\vec{v}_2)$

$= \lambda\vec{v} \Rightarrow \vec{v}$ an eigenvector

(if non-zero)

(b) The first column of A is $A\begin{pmatrix}1\\0\end{pmatrix} = \lambda\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\lambda\\0\end{pmatrix}$ since $\begin{pmatrix}1\\0\end{pmatrix}$ is an eigenvector by (a). Similarly, the 2nd column of A is $A\begin{pmatrix}0\\1\end{pmatrix} = \lambda\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\\lambda\end{pmatrix}$. Thus

$$A = \begin{pmatrix}\lambda & 0\\0 & \lambda\end{pmatrix}.$$

(38a) $\vec{V} = \begin{pmatrix}3+5i\\4\end{pmatrix} = \begin{pmatrix}3\\4\end{pmatrix} + i\begin{pmatrix}5\\0\end{pmatrix}$ $\quad \vec{a} = \begin{pmatrix}3\\4\end{pmatrix}, \vec{b} = \begin{pmatrix}5\\0\end{pmatrix}$

If z is a complex number, let $\tilde{a}, \tilde{b}$ be the real + imaginary parts of $z\cdot\vec{V}$. That is $z\vec{V} = \tilde{a} + i\tilde{b}$.

If $z = \alpha + i\beta$ then

$$z\vec{v} = (\alpha+i\beta)\left[\begin{pmatrix}3\\4\end{pmatrix} + i\begin{pmatrix}5\\0\end{pmatrix}\right] = \begin{pmatrix}3\alpha\\4\alpha\end{pmatrix} - \begin{pmatrix}5\beta\\0\end{pmatrix} + i\left[\begin{pmatrix}3\beta\\4\beta\end{pmatrix} + \begin{pmatrix}5\alpha\\0\end{pmatrix}\right]$$

$$= \begin{pmatrix}3\alpha-5\beta\\4\alpha\end{pmatrix} + i\begin{pmatrix}3\beta+5\alpha\\4\beta\end{pmatrix} \qquad \tilde{a} = \begin{pmatrix}3\alpha-5\beta\\4\alpha\end{pmatrix}, \tilde{b} = \begin{pmatrix}3\beta+5\alpha\\4\beta\end{pmatrix}$$

$$\tilde{a}\cdot\tilde{b} = 0 \Leftrightarrow \begin{pmatrix}3\alpha-5\beta\\4\alpha\end{pmatrix}\cdot\begin{pmatrix}3\beta+5\alpha\\4\beta\end{pmatrix} = 0$$

$$\Leftrightarrow (3\alpha-5\beta)(3\beta+5\alpha) + 16\alpha\beta = 0$$

$$\Leftrightarrow 15\alpha^2 - 15\beta^2 + \cancel{9\alpha\beta} - \cancel{25\alpha\beta} + \cancel{16\alpha\beta} = 0$$

$$\Leftrightarrow 15(\alpha^2-\beta^2) = 0 \quad \Leftrightarrow \alpha = \pm\beta \Leftrightarrow z = r(1\pm i) \text{ some } r=\alpha=\beta$$

Sec. 6.3

(4) $\frac{dx}{dt} = 60x - 4x^2 - 3xy$

$\frac{dy}{dt} = 42y - 2y^2 - 3xy$

Jacobian: $D\vec{F}(x,y) = \begin{pmatrix} 60-8x-3y & -3x \\ -3y & 42-4y-3x \end{pmatrix}$

$D\vec{F}(0,0) = \begin{pmatrix} 60 & 0 \\ 0 & 42 \end{pmatrix}$ $\lambda_1 = 60$, $\lambda_2 = 42$ both positive

$\Rightarrow$ $(0,0)$ unstable node

(5) At critical point $(0,21)$

$D\vec{F}(0,21) = \begin{pmatrix} 60-63 & 0 \\ -63 & 42-84 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ -63 & -42 \end{pmatrix}$

Thus the linearized eq. is

$\vec{y}\,' = \begin{pmatrix} -3 & 0 \\ -63 & -42 \end{pmatrix} \vec{y}$ or if $\vec{y} = \begin{pmatrix} u \\ v \end{pmatrix}$

$u' = -3u$

$v' = -63u - 42v$

Eigenvalues are the diagonal entries of $D\vec{F}(0,21)$ since it is a triangular matrix. Thus $\lambda_1 = -3$, $\lambda_2 = -42$. Both negative so $(0,21)$ is a stable node.

(9) $\dfrac{dx}{dt} = 60x - 3x^2 - 4xy$

$\dfrac{dy}{dt} = 42y - 3y^2 - 2xy$

$$D\vec{F}(x,y) = \begin{pmatrix} 60-6x-4y & -4x \\ -2y & 42-6y-2x \end{pmatrix}$$

At $(20,0)$ $\quad D\vec{F}(20,0) = \begin{pmatrix} 60-120-0 & -80 \\ 0 & 42-40 \end{pmatrix} = \begin{pmatrix} -60 & -80 \\ 0 & 2 \end{pmatrix}$

Linearization: $\quad \begin{pmatrix} u \\ v \end{pmatrix}' = DF(20,0)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -60 & -80 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$\Rightarrow \quad u' = -60u - 80v$

$\quad\quad v' = 2v$

Eigenvalues are the diagonal entries as $DF(20,0)$ is upper triangular

$\Rightarrow \lambda_1 = -60, \ \lambda_2 = 2$. Of opposite sign $\Rightarrow (20,0)$

is an <u>unstable saddle point</u>.