Homework 5 Solutions

Sec. 4.1

(2) $x'' + 4x - x^3 = 0$

Let $y = x'$. Then we get $y' = x''$, $x'' = x^3 - 4x$

$$\Rightarrow \boxed{\begin{aligned} x' &= y \\ y' &= x^3 - 4x \end{aligned}}$$

* (6) $x^{(4)} + 6x'' - 3x' + x = \cos 3t$

Let $x_1 = x$, $x_2 = x'$, $x_3 = x''$, $x_4 = x'''$. Then

$x_4' = x^{(4)} = -6x'' + 3x' - x + \cos 3t \Rightarrow$

$$\begin{cases} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ x_4' = -x_1 + 3x_2 - 6x_3 + \cos 3t \end{cases}$$

(13) $x'' = -75x + 25y$

$y'' = 50x - 50y + 50\cos 5t$

Let $x_1 = x$, $x_2 = x'$, $\cancel{x}\, y_1 = y$, $y_2 = y'$ (could use x_3, x_4 in place of y_1, y_2)

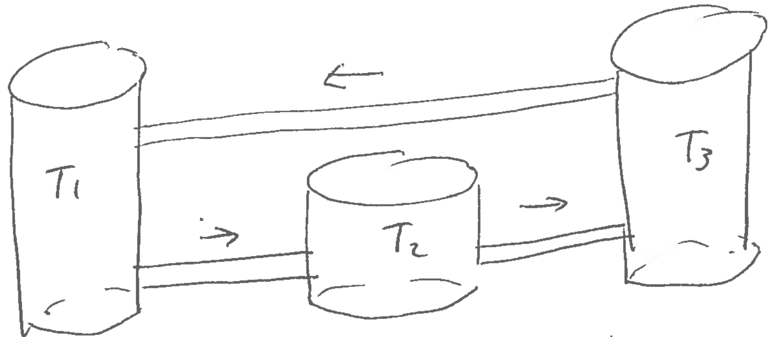
then get

$$\begin{cases} x_1' = x_2 \\ x_2' = -75x_1 + 25y_1 \\ y_1' = y_2 \\ y_2' = 50x_1 - 50y_1 + 50\cos 5t \end{cases}$$

(32) x_i = ammount in tank T_i

volume rates circulate between tanks at 10 gal./min.

Volumes of each tank are 100 gal.



concentration c_i of ith tank is $\dfrac{x_i}{100}$

rate of ammount of solute leaving ith tank is $r c_i$

or $$r \frac{x_i}{100} = \frac{10\, x_i}{100} = \frac{x_i}{10}.$$

$$x_1' = r c_3 - r c_1 = \frac{x_3}{10} - \frac{x_1}{10}$$

$$x_2' = r c_1 - r c_2 = \frac{x_1}{10} - \frac{x_2}{10}$$

$$x_3' = r c_2 - r c_3 = \frac{x_2}{10} - \frac{x_3}{10}$$

multiply each eq. by 10:

$$\Rightarrow \begin{aligned} 10 x_1' &= -x_1 + x_3 \\ 10 x_2' &= x_1 - x_2 \\ 10 x_3' &= x_2 - x_3 \end{aligned}$$

Sec. 4.3 (2)

$$x' = 2x + 3y \qquad x(0) = 1$$
$$y' = 2x + y \qquad y(0) = -1$$

$h = 0.1$

$t_0 = 0,\ t_1 = 0.1,\ t_2 = 0.2$

In vector form this is

$$\vec{x}\,' = F(\vec{x}) \qquad \vec{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ 2x + y \end{pmatrix}$$

* (a) For Euler!

$$\vec{x}_{n+1} = \vec{x}_n + hF(\vec{x}_n) \qquad \vec{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So $\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0.1 \begin{pmatrix} 2-3 \\ 2-1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0.1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.9 \\ -0.9 \end{pmatrix}$

$$\vec{x}_2 = \begin{pmatrix} 0.9 \\ -0.9 \end{pmatrix} + 0.1 \begin{pmatrix} 2(0.9) + 3(-0.9) \\ 2(0.9) - 0.9 \end{pmatrix} = \begin{pmatrix} 0.9 \\ -0.9 \end{pmatrix} + 0.1 \begin{pmatrix} -0.9 \\ 0.9 \end{pmatrix} = \begin{pmatrix} 0.81 \\ -0.81 \end{pmatrix}$$

Approximate values at 0.2:
from Euler

$$\boxed{\begin{aligned} x(0.2) &\approx 0.81 \\ y(0.2) &\approx -0.81 \end{aligned}}$$

True Values:

$$\boxed{\begin{aligned} x(0.2) &= e^{-0.2} = 0.8187 \\ y(0.2) &= -e^{-0.2} = -0.8187 \end{aligned}}$$

b. Improved Euler:

$$\vec{u}_{n+1} = \vec{x}_n + h\,\vec{F}(\vec{x}_n)$$

$$\vec{x}_{n+1} = \vec{x}_n + \frac{h}{2}\left(\vec{F}(\vec{x}_n) + \vec{F}(\vec{u}_{n+1})\right)$$

$$\vec{x}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\vec{u}_1 = \vec{x}_1$ from Euler Method in (a) $= \begin{pmatrix} 0.9 \\ -0.9 \end{pmatrix}$

$$\vec{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{0.1}{2}\left(\begin{pmatrix} -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2(0.9) + 3(-0.9) \\ 2(0.9) - 0.9 \end{pmatrix}\right)$$

$$= \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 0.05 \begin{pmatrix} -1.9 \\ 1.9 \end{pmatrix} = \begin{pmatrix} 0.905 \\ -0.905 \end{pmatrix}$$

$$\vec{u}_2 = \begin{pmatrix} 0.905 \\ -0.905 \end{pmatrix} + 0.1 \begin{pmatrix} 2(0.905) + 3(-0.905) \\ 2(0.905) - 0.905 \end{pmatrix} = \begin{pmatrix} 0.905 \\ -0.905 \end{pmatrix} + \begin{pmatrix} -0.0905 \\ 0.0905 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8145 \\ -0.8145 \end{pmatrix}$$

$$\vec{x}_2 = \begin{pmatrix} 0.905 \\ -0.905 \end{pmatrix} + \frac{0.1}{2}\left(\begin{pmatrix} 2(0.905)-3(0.905) \\ 2(0.905)-(0.905) \end{pmatrix} + \begin{pmatrix} 2(0.8145)-3(0.8145) \\ 2(0.8415)-0.8415 \end{pmatrix} \right)$$

$$= \begin{pmatrix} 0.905 \\ -0.905 \end{pmatrix} + 0.05 \begin{pmatrix} -1.7195 \\ 1.7195 \end{pmatrix} = \begin{pmatrix} 0.8190 \\ -0.8190 \end{pmatrix}$$

Approx. with Improved Euler at 0.2:

$$\boxed{\begin{aligned} x(0.2) &\approx 0.8190 \\ y(0.2) &\approx -0.8190 \end{aligned}}$$

compared with true value:

$$\boxed{\begin{aligned} x(0) &= 0.8187 \\ y(0) &= -0.8187 \end{aligned}}$$

Sec 5.1

(13) $x' = 2x + 4y + 3e^t, \quad y' = 5x - y - t^2$

$$\vec{x}' = \begin{pmatrix} 2 & 4 \\ 5 & -1 \end{pmatrix} \vec{x} + \begin{pmatrix} 3e^t \\ -t^2 \end{pmatrix}$$

(15)

$$\begin{aligned} x' &= y + z \\ y' &= z + x \\ z' &= x + y \end{aligned} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$\underline{\text{or}}$ $\quad \vec{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \vec{x}$

* (20)

$$\begin{aligned} x_1' &= x_2 + x_3 + 1 \\ x_2' &= x_3 + x_4 + t \\ x_3' &= x_1 + x_4 + t^2 \\ x_4' &= x_1 + x_2 + t^3 \end{aligned} \Rightarrow \boxed{\vec{x}' = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} 1 \\ t \\ t^2 \\ t^3 \end{pmatrix}}$$

(26) $\vec{x}\,' = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix} \vec{x}$ $\quad \vec{x}_1 = e^{t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}, \vec{x}_2 = e^{3t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$\vec{x}_3 = e^{5t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$\vec{x}_1\,' = e^{t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \quad A\vec{x}_1 = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix} e^{t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

$= e^{t}\begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = e^{t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = \vec{x}_1\,'$

$\vec{x}_2\,' = 3e^{t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

$A\vec{x}_2 = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix} e^{3t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = e^{3t}\begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = e^{3t}\begin{pmatrix} -6 \\ 0 \\ 3 \end{pmatrix}$

$= 3e^{3t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = \vec{x}_2\,'$ (factor out 3)

$\vec{x}_3\,' = 5e^{5t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad A\vec{x}_3 = \begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix} e^{5t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = e^{5t}\begin{pmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$= e^{5t}\begin{pmatrix} 10 \\ -10 \\ 5 \end{pmatrix} = 5e^{5t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \vec{x}_3\,'.$

$W(t) = \det\left(\vec{x}_1(t) \;\; \vec{x}_2(t) \;\; \vec{x}_3(t)\right)$. By Theorem 2, it suffices to show $\det\left(\vec{x}_1(0), \vec{x}_2(0), \vec{x}_3(0)\right) \neq 0$.

(though you could do for all t).

$$W(0) = \det\begin{pmatrix} 2 & -2 & 2 \\ 2 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} = 2\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & \cancel{2} \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix}$$

$$= 2\cdot 2 + 2\cdot 4 - 2\cdot 2 = 8 \neq 0.$$

(35.)* Solve above system with $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 4$.

The solution is of form $\vec{x}(t) = C_1\vec{x}_1(t) + C_2\vec{x}_2(t) + C_3\vec{x}_3(t)$

We need $\vec{x}(0) = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} \Leftrightarrow \begin{pmatrix} \vec{x}_1(0) & \vec{x}_2(0) & \vec{x}_3(0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

i.e. we must solve

$$\left(\begin{array}{ccc|c} 2 & -2 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 1 & 1 & 4 \end{array}\right)$$

By Gauss-Jordan $\Rightarrow$ $\left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 1 & 1 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & 0 & 4 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \Rightarrow \begin{array}{l} c_1 = 1 \\ c_2 = 2 \\ c_3 = 1 \end{array}$$

$$\Rightarrow \boxed{\vec{x}(t) = e^{t}\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + 2e^{3t}\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + e^{5t}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}$$

Sec. 5.2 * (2) $x_1' = 2x_1 + 3x_2$, $x_2' = 2x_1 + x_2$

$$\Rightarrow \vec{x}' = \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \vec{x}$$

eigenvalues: $\det\begin{pmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{pmatrix} = (2-\lambda)(1-\lambda) - 6 = 0$

$$\lambda^2 - 3\lambda - 4 = 0 \Rightarrow (\lambda-4)(\lambda+1) = 0, \quad \boxed{\lambda = -1, 4}$$

Eigenvectors: $\lambda_1 = -1$: $\left(\begin{array}{cc|c} 3 & 3 & 0 \\ 2 & 2 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = -x_2$, setting $x_2 = 1$ gives eigenvector $\vec{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda_2 = 4$: $\left(\begin{array}{cc|c} -2 & 3 & 0 \\ 2 & -3 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -3/2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = \frac{3}{2}x_2$. Setting $x_2 = 2$ gives $x_1 = 3$ and

eigenvector $\vec{v}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$\Rightarrow$ general soln. is $\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$

$$\boxed{\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}}$$

Any multiple of these eigenvectors also gives general soln.

(5) $x_1' = 6x_1 - 7x_2$
$x_2' = x_1 - 2x_2$ $\Rightarrow$ $\vec{x}' = \begin{pmatrix} 6 & -7 \\ 1 & -2 \end{pmatrix} \vec{x}$

Eigenvalues: $\begin{vmatrix} 6-\lambda & -7 \\ 1 & -2-\lambda \end{vmatrix} = (6-\lambda)(-2-\lambda) + 7 = 0$

$\lambda^2 - 4\lambda - 5 = 0 \Rightarrow (\lambda-5)(\lambda+1) = 0$

$\lambda_1 = -1, \lambda_2 = 5$

Eigenvectors: $\lambda_1 = -1$: $\left(\begin{array}{cc|c} 7 & -7 & 0 \\ 1 & -1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = x_2 \Rightarrow \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
set $x_2 = 1$

$\lambda_2 = 5$: $\left(\begin{array}{cc|c} 1 & -7 & 0 \\ 1 & -7 & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -7 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$x_1 = 7x_2$, set $x_2 = 1 \Rightarrow \vec{v}_2 = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$

General soln: $\boxed{\vec{x}(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{5t}\begin{pmatrix} 7 \\ 1 \end{pmatrix}}$

(11) $x_1' = x_1 - 2x_2$ $\qquad x_1(0) = 0$

* $x_2' = 2x_1 + x_2$ $\qquad x_2(0) = 4$

$$\vec{x}' = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

Eigenvalues: $\begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 4 = 0$

$$\lambda^2 - 2\lambda + 5 = 0 \qquad \lambda = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm \frac{4i}{2} = 1 \pm 2i$$

take eigenvalue $\lambda = 1+2i$, find complex eigenvector:

$$\left(\begin{array}{cc|c} -2i & -2 & 0 \\ 2 & -2i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -i & 0 \\ 2 & -2i & 0 \end{array}\right) \rightarrow \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right)$$

$x_1 = i x_2$. Setting $x_2 = 1$ gives

$$\vec{v} = \begin{pmatrix} i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Complex soln: $e^{\lambda t}\vec{v} = e^{t}(\cos 2t + i \sin 2t)\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$

$$= e^{t}\left(\cos 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix} - \sin 2t\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) + i\, e^{t}\left(\cos 2t\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sin 2t\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

take real + imaginary parts to get basis:

$$\vec{x}_1 = e^{t}\left(\cos 2t \begin{pmatrix}0\\1\end{pmatrix} - \sin 2t \begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$x_2 = e^{t}\left(\cos 2t \begin{pmatrix}1\\0\end{pmatrix} + \sin 2t \begin{pmatrix}0\\1\end{pmatrix}\right)$$

Solving for initial condition:

$$\vec{x}(0) = c_1 \vec{x}_1(0) + c_2 \vec{x}_2(0) = \begin{pmatrix}0\\4\end{pmatrix} \qquad \vec{x}_1(0) = \begin{pmatrix}0\\1\end{pmatrix}$$

$$\vec{x}_2(0) = \begin{pmatrix}1\\0\end{pmatrix}$$

$$\Rightarrow \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 1 & 0 & 4 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 0 \end{array}\right)$$

$$\Rightarrow c_2 = 0, \quad c_1 = 4$$

$$\Rightarrow \boxed{\vec{x}(t) = 4e^{t}\left(\cos 2t \begin{pmatrix}0\\1\end{pmatrix} - \sin 2t \begin{pmatrix}1\\0\end{pmatrix}\right)}$$

Of course this can be written: $\boxed{\begin{aligned} x_1(t) &= -4e^{t} \sin 2t \\ x_2(t) &= 4e^{t} \cos 2t. \end{aligned}}$

(23) $$\begin{aligned} x_1' &= 3x_1 + x_2 + x_3 \\ x_2' &= -5x_1 - 3x_2 - x_3 \\ x_3' &= 5x_1 + 5x_2 + 3x_3 \end{aligned} \Rightarrow \vec{x}' = \begin{pmatrix} 3 & 1 & 1 \\ -5 & -3 & -1 \\ 5 & 5 & 3 \end{pmatrix} \vec{x}$$

Eigenvalues: $$\begin{vmatrix} 3-\lambda & 1 & 1 \\ -5 & -3-\lambda & -1 \\ 5 & 5 & 3-\lambda \end{vmatrix} = (3-\lambda)\left[(-3-\lambda)(3-\lambda) + 5\right]$$

$$- \left[-5(3-\lambda) + 5\right] + \left[-25 - 5(-3-\lambda)\right]$$

$$= 0$$