Homework 2 Solutions

Sec. 2.1 (15) $\frac{dP}{dt} = aP - bP^2$ $\quad B = aP$ = birth rate per unit time

$P(0) = P_0$ $\quad D = bP^2$ = death rate per unit time

B_0 = Birth rate at $t=0$

D_0 = Death rate at $t=0$

Putting the logistic equation in the form

$$\frac{dP}{dt} = bP\left(a/b - P\right) = kP(M-P)$$ we know that

the limiting population is $M = a/b$ from 2.1.

However $B_0 = aP_0$ and $D_0 = bP_0^2$ from

so $a = B_0/P_0$ and $b = D_0/P_0^2$

$$\Rightarrow M = a/b = \frac{B_0/P_0}{D_0/P_0^2} = \frac{B_0 P_0}{D_0}$$

(21) $\frac{dP}{dt} = kP(200-P)$ Set $t=0 \Leftrightarrow 1960$

When $t=0$ $\quad P_0 = 150$ and $\frac{dP}{dt} = 1$, given

so to solve for k use the equation

$$1 = k \cdot 100\,(200-100) \Rightarrow k = 1/10{,}000$$

The solution is then (from (7) in text)

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}} \qquad M = 200$$

$$\Rightarrow P(t) = \frac{20{,}000}{100 + 100e^{\frac{-200}{10{,}000}t}}$$

So 2020 : $t = 60$

$$P(60) = \frac{20{,}000}{100 + e^{\frac{-3}{100}\cdot(60)}} = \frac{20{,}000}{100 + e^{-1.2}} = \boxed{153.7}$$

(29) $\frac{dP}{dt} = 0.03135P - 0.0001489P^2$

$P(0) = 3.9 \qquad t = 0 : 1790$

$$\frac{dP}{dt} = 0.0001489P\left(\frac{0.03135}{0.0001489} - P\right) = kP(M-P)$$

$k = 0.0001489 \qquad M = \frac{0.03135}{0.0001489} = 210.5 \qquad P_0 = 3.9$

$$P(t) = \frac{MP_0}{P_0 + (M-P_0)e^{-kMt}} = \frac{820.95}{3.9 + 206.6e^{-.0314t}}$$

(a) Predicted value for 1930 : $t = 1930 - 1790 = 140$

$$P(140) = \frac{820.95}{3.9 + 206.6e^{-0.0314\cdot(140)}} = \boxed{127.3}$$

(b) Limiting population predicted is $M = 210.5$ million (from above)

(c) No, considering by now the population is around 300 million (a fair amount bigger than 210.5).

Sec. 2.2

(9) $\frac{dx}{dt} = x^2 - 5x + 4$

$f(x) = x^2 - 5x + 4$. Critical points $f(x) = 0$

$\Rightarrow (x-4)(x-1) = 0 \Rightarrow \left.\begin{array}{l} x = 4 \\ x = 1 \end{array}\right\}$ constant solutions.

Note $f(x) = (x-4)(x-1)$

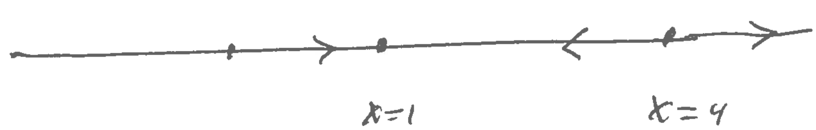
so if $x > 4 \Rightarrow f(x) > 0$ soln's increase away from 4 ~~decrease to x=4~~

$1 < x < 4 \Rightarrow f(x) < 0$ solutions decrease away from 4 towards 1

$x < 1$ $f(x) > 0 \Rightarrow$ soln's increase towards 1.

So $\boxed{x = 4 \text{ unstable}}$

$\boxed{x = 1 \text{ is asymptotically stable}}$



Solving:

$$\frac{dx}{(x-4)(x-1)} = dt \qquad \int \frac{dx}{(x-4)(x-1)} = \int dt + C$$

$$\frac{1}{(x-4)(x-1)} = \frac{A}{x-4} + \frac{B}{x-1} \Rightarrow 1 = A(x-1) + B(x-4)$$

$x=1 \Rightarrow 1 = -3B \Rightarrow B = -1/3$

$x=4 \Rightarrow 1 = 3A \Rightarrow A = 1/3$

$$\frac{1}{3}\int \frac{dx}{x-4} - \frac{1}{3}\int \frac{dx}{x-1} = t + C \qquad C \text{ const.}$$

$$\frac{1}{3}\ln|x-4| - \frac{1}{3}\ln|x-1| = t + c$$

$$\ln|x-4| - \ln|x-1| = 3t + C \qquad \text{(diffent C)}$$

$$\ln\left|\frac{x-4}{x-1}\right| = 3t + c \qquad \text{exponentiate}$$

$$\left|\frac{x-4}{x-1}\right| = e^{3t+c} = e^c e^{3t}$$

$$\Rightarrow \frac{x-4}{x-1} = Ke^{3t} \qquad K = \pm e^c.$$

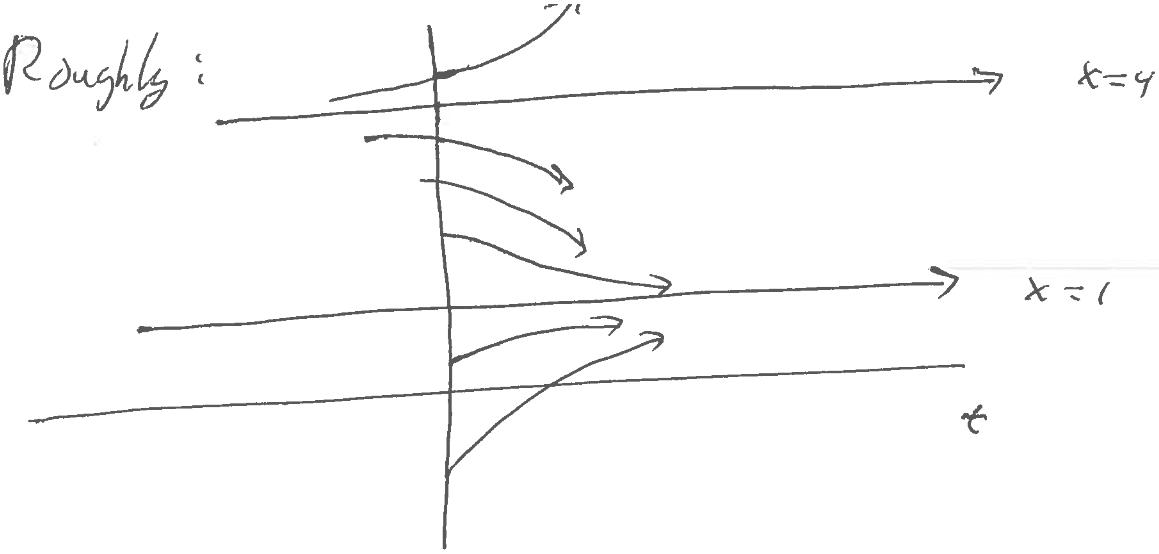
$$\Rightarrow x - 4 = (x-1)Ke^{3t} = xKe^{3t} - Ke^{3t}$$

$$x - xKe^{3t} = 4 - Ke^{3t} \Rightarrow x(1 - Ke^{3t}) = 4 - Ke^{3t}$$

$$\boxed{x = \frac{4 - Ke^{3t}}{1 - Ke^{3t}}}$$

$K =$ const.

Roughly:



(17) $\frac{dx}{dt} = x^2(x^2-4)$

We'll just use theorems in class to determine stability of critical points:

$f(x) = x^2(x^2-4) = x^2(x+2)(x-2)$

$f(x) = 0 \iff \underbrace{x=0,\ x=-2,\ x=2}$ critical points.

~~If $x < -2$, $x^2-4<0$, $x^2>0$ $\Rightarrow$ $f(x)<0$~~
~~$-2 < x < 0$ $x^2-4>0$, $x^2>0$ $\Rightarrow$ $f(x)>0$~~
~~$0 < x < 2$ $x^2-4<0$ $\Rightarrow$ $f(x)$~~

If $x<-2$, $x^2-4>0$, $x^2>0$ $\Rightarrow$ $f(x)>0$

$-2<x<0$ $x^2-4<0$ $\Rightarrow$ $f(x)<0$

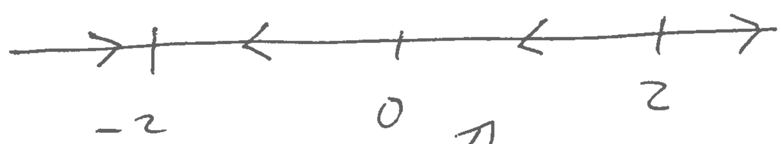
$0<x<2$ $x^2-4<0$ $\Rightarrow$ $f(x)<0$

$2<x$ $x^2-4>0$ $\Rightarrow$ $f(x)>0$

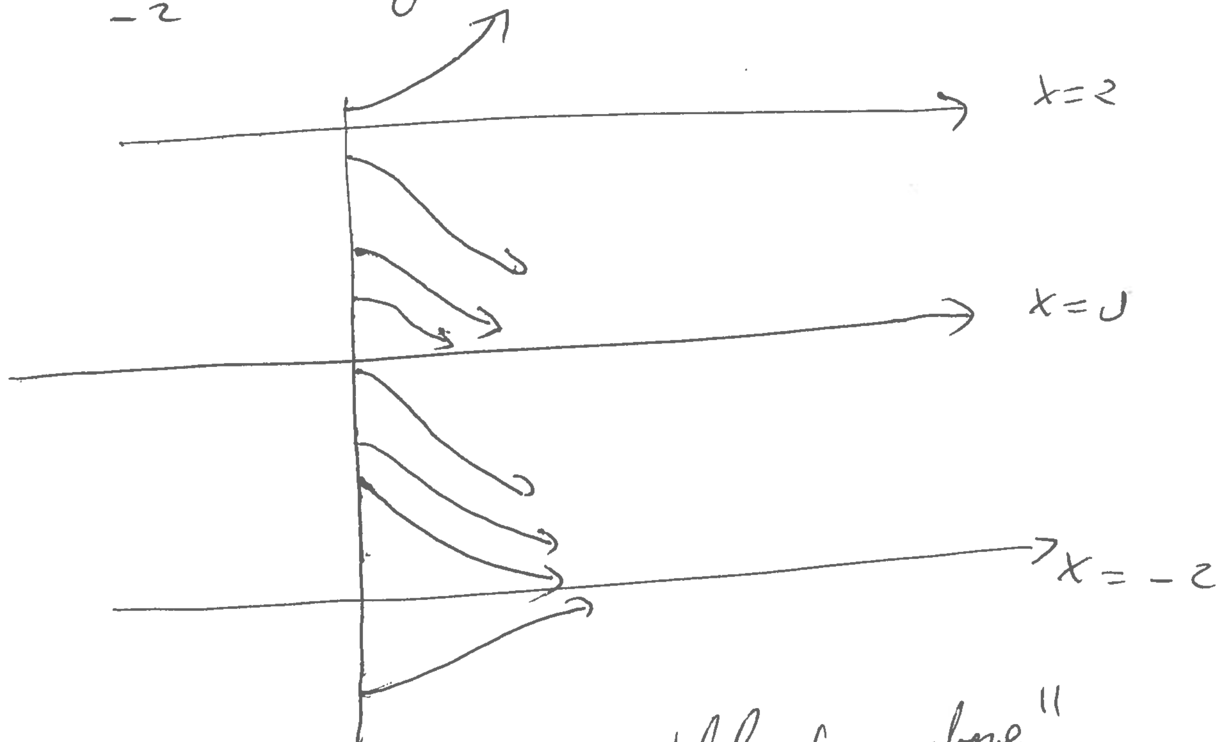
$\Rightarrow$ -2 is asymtotically stable (Can also check that $f'(-2)<0$ to show -2 is stable

$\Rightarrow$ 0 is unstable

2 is unstable — $f'(2)>0$ so unstable.



phase diagram.



Note that $x=0$ is "stable from above"!

(21) $\dfrac{dx}{dt} = kx - x^3$ $\qquad k$ const.

(a) If $k \leq 0 \Rightarrow f(x) = kx - x^3 = x(k - x^2)$

if $k=0$ then only $x=0$ is a critical point.

if $k<0 \Rightarrow k - x^2 < 0$ all $x \Rightarrow$ only $x=0$ critical point.

In either case, if $x < 0$, $k - x^2 < 0$

$\Rightarrow f(x) > 0$

If $x > 0$ $k - x^2 < 0 \Rightarrow f(x) < 0$

$\Rightarrow$ 0 is asymtotically stable.

(b) If $k > 0$ $f(x) = x(k - x^2) = x(\sqrt{k} - x)(\sqrt{k} + x)$

$\Rightarrow$ critical points $x = 0$, $x = -\sqrt{k}$, $x = +\sqrt{k}$

In this case:

i) $x < -\sqrt{k} \Rightarrow k - x^2 < 0,\ x < 0 \Rightarrow f(x) > 0$

ii) $-\sqrt{k} < x < 0 \Rightarrow k - x^2 > 0,\ x < 0 \Rightarrow f(x) < 0$

iii) $0 < x < \sqrt{k}$, $k - x^2 > 0,\ x > 0 \Rightarrow f(x) > 0$

iv) $\sqrt{k} < x$, $k - x^2 < 0,\ x > 0 \Rightarrow f(x) < 0$

From i), ii) $\Rightarrow$ $-\sqrt{k}$ is stable

From ii), iii) $\Rightarrow$ 0 is unstable

From iii), iv) $\Rightarrow$ $\sqrt{k}$ is stable

2.31 (2) $\dfrac{dv}{dt} = -kv$ Separable so

(a) $\dfrac{dv}{v} = -k\,dt \Rightarrow \displaystyle\int \frac{dv}{v} = -k\int dt + C$

$\ln|v| = -kt + c$ C const.

$\Rightarrow |v| = e^{-kt+c} = e^{c}e^{-kt}$

$\Rightarrow v = \pm e^{c}e^{-kt} \Rightarrow v = Ke^{-kt}$

$v(0) = K = v_0 \Rightarrow \boxed{v(t) = v_0 e^{-kt}}$

(b) $x(t) = \displaystyle\int v(t)\,dt + C = v_0 \int e^{-kt}\,dt + C$

$x(t) = -\dfrac{v_0}{k} e^{-kt} + C$

$x(0) = x_0 = -\dfrac{v_0}{k} + C \Rightarrow C = x_0 + \dfrac{v_0}{k}$

$\Rightarrow x(t) = x_0 + \dfrac{v_0}{k} - \dfrac{v_0}{k} e^{-kt} < x_0 + \dfrac{v_0}{k}$

Farthest can go $\cancel{\lim_{t\to\infty} x(t) = x_0 + \frac{v_0}{k}}$

from time 0, $x = x_0$ is then $x_0 + \dfrac{v_0}{k}$ so

dist. traveled is $\boxed{v_0/k}$

(3) Assume $\frac{dv}{dt} = -kv$. From problem 2

$v(t) = v_0 e^{-kt}$

Given $v_0 = 40$ f/s (when motor quits)

10 s later $v(10) = 40 e^{-k \cdot 10} = 20$

Solve for k: $e^{-k10} = \frac{1}{2} \Rightarrow -k10 = \ln(\frac{1}{2}) = -\ln 2$

$\Rightarrow \boxed{k = \frac{\ln 2}{10}}$

From problem 2 it will coast as far as $+\frac{v_0}{k}$

or $\frac{40}{\ln 2/10} = \boxed{\frac{400}{\ln 2} \text{ or } 577 \text{ f.}}$

(11) Assuming a terminal velocity $v_\tau = -100$ mi/h

Converting to f/s: $v_\tau = -146.7$ f/s

$|v_\tau| = \frac{g}{\rho} \Rightarrow \rho = g/|v_\tau| = \frac{32}{146.7} = 0.22$

From (9) in text, height at time t

$y_0 = 1200$
$v_0 = 0$

$y(t) = y_0 + v_\tau t - \frac{1}{\rho}(v_0 - v_\tau)(1 - e^{-\rho t}) \Rightarrow$

$$y(t) = 1200 - 146.7t - \frac{1}{0.22}(0 + 146.7)(1 - e^{-0.22t})$$

So if terminal velocity was 100 miles/hr = 146.7 f/s then the time it would take to fall would be the solution of $y(t) = 0$ or

$$0 = 1200 - 146.7t - 666.8(1 - e^{-0.22t})$$

This is not solvable analytically (I think?). Howev If one plugs in $t = 8$, one is not close to a soln. In fact $y(8)$ is several hundred feet, so $t > 8$. According to text $t = 12.5$. (Can be solved with a computer program e.g. Newton's Method).

(29) ~~In class we derived the formula in general~~

$$\cancel{v^2 = v_0^2 + \frac{2MG}{y} - \frac{2MG}{R_0}}$$

~~by using the suggestion in text. Putting over common denominator:~~

$$\cancel{v^2 = v_0^2 + \frac{2MGR}{Ry} - \frac{2MGy}{Ry} =}$$

(29) $$\frac{d^2y}{dt^2} = \frac{-GM}{(y+R)^2} \qquad y(0)=0\,,\quad y'(0)=v_0$$

$v = \dfrac{dy}{dt}$ so by the chain rule (using $y=y(t)$)

$$\frac{dv}{dt} = \frac{dv}{d\cancel{y}}\frac{d\cancel{y}}{dt} = v\frac{dv}{dy}$$

$$\Rightarrow v\frac{dv}{dy} = \frac{-GM}{(y+R)^2}$$

$$v\,dv = -GM\frac{dy}{(y+R)^2}$$

$$\int v\,dv = -GM\int\frac{dy}{(y+R)^2} + C$$

$$\frac{v^2}{2} = -GM\cdot\left(-(y+R)^{-1}\right) + C$$

$$v^2 = \frac{-2GM}{y+R} + C$$

$$v_0^2 = \frac{-2GM}{R} + C \Rightarrow C = v_0^2 + \frac{2GM}{R}$$

$$\rightarrow v^2 = v_0^2 + \frac{2GM}{R} - \frac{2GM}{y+R}$$

Put over common denominator:

$$v^2 = v_0^2 \pm \frac{2GM(y+R) - 2GMR}{R(y+R)}$$

$$\boxed{v^2 = v_0^2 \pm \frac{2GMy}{R(y+R)}}$$

For earth $R = 6.378 \times 10^6$ m

$M = 5.975 \times 10^{24}$ kg.

$v_0 = 1$ km/s $= 1000$ m/s

Max height will be when $v = 0$

$\Rightarrow v_0^2 = \dfrac{2GMy}{R(y+R)}$ Solve for y:

$$v_0^2 R(y+R) = 2GMy$$

$$v_0^2 Ry + v_0^2 R^2 = 2GMy \Rightarrow (2GM - v_0^2 R)\, y = v_0^2 R^2$$

$$y = \frac{v_0^2 R^2}{2GM - v_0^2 R}$$

$G = 6.6726 \times 10^{-11}$

or $$\frac{(6.378 \times 10^6)^2 \times 10^6}{2 \times 6.6726 \times 10^{-11} \times 5.975 \times 10^{24} - (6.378 \times 10^6) \times 10^6} = 51{,}400\text{ m}$$

or 51.4 km

Sec. 2.4

(26) $\dfrac{dP}{dt} = 0.0225P - 0.0003P^2$

$P(0) = 27$

Step Size $h=1$, Eulers Method :

$$f(t,P) = f(P) = 0.0225P - 0.0003P^2$$

Interval $[0,10]$. So $x_k = k$, $k = 0, \dots, 10$

$$P_{k+1} = P_k + hf(P_k) \quad \text{or}$$

$$\boxed{\begin{array}{l} P_{k+1} = P_k + (0.0225P_k - 0.0003P_k^2) \\ P_0 = 25 \end{array}}$$

$$\cancel{\begin{array}{l} P_1 = 25 + (.0225 \cdot 25 - .0003(25)^2) = 25.375 \\ P_2 = 25.375 + ((.0225)\cdot(25.375) - .0003(25.375)^2) \end{array}}$$

this can be simplified to

$$\boxed{P_{k+1} = 1.0225P_k - 0.0003P_k^2}$$

$$P_1 = 1.0225 \cdot 25 - 0.0003(25)^2 = 25.375$$

$$P_2 = (1.0225)\cdot(25.375) - (.0003)(25.375)^2 = 25.76$$

$$P_3 = (1.0225)\cdot(25.76) - (0.0003)(25.76)^2 = 26.14$$

$$P_4 = (1.0225)(26.14) - (.0003)(26.14)^2 = 26.53$$

$$P_5 = (1.0225)(26.53) - (0.0003)(26.53)^2 = 26.92$$

$$P_6 = (1.0225)(26.92) - (0.0003)(26.92)^2 = 27.31$$

$$P_7 = (1.0225)(27.31) - (0.0003)(27.31)^2 = 27.70$$

$$P_8 = (1.0225)(27.7) - (0.0003)(27.7)^2 = 28.09$$

$$P_9 = (1.0225)(28.09) - (0.0003)(28.09)^2 = 28.49$$

$$P_{10} = (1.0225)(28.49) - (0.0003)(28.49)^2 = 28.89$$

~~So rounded to a whole number, this predicts a population of 29 deer after 10 years.~~

~~This is $\frac{29}{75} \times 100$ or 38.7% of the limiting population of 75 deer.~~

$P_{11} = 29.29$

$P_{12} = 29.69$ (after 1 year)

For 10 years, can use a programmable calculator or computer.