Sample Quiz 9

Extra Credit Problem 1. Piecewise Continuous Inputs

Consider a passenger SUV on a one-day trip from Salt Lake City to Pine Bluffs, Wyoming, on the Nebraska border. The route is I-80 E, 471 miles through Utah and Wyoming. Google maps estimates 6 hours and 52 minutes hours driving time. The table below shows the distances, time, road segment and average speed with total trip time 7 hours and 42 minutes. Cities enroute reduce the freeway speed by 10 mph, the trip time effect not shown in the table.

Miles	Minutes	Speed mph	Road Segment	Posted limit mph
18.1	20	54.3	Parley's Walmart to Kimball	65
11.3	12	56.3	Kimball to Wanship	$65 - 55$
9.1	13	42	Wanship to Coalville	70
5.7	7	48.9	Coalville to Echo Dam	70
16.5	18	55	Echo Dam to 75 mph sign	70
39	36	65	75 mph sign to Evanston	75
308	269	68.7	Evanston to Laramie	75
50.6	50	61.2	Laramie to Cheyenne	75
43	37	69.7	Cheyenne to Pine bluffs	75

The **velocity function** for the SUV is approximated by

$$
V_{\text{pc}}(t) = \begin{cases}
\begin{array}{ccc}
\text{Speed mph} & \text{Time interval minutes} & \text{Road segment} \\[4pt]
54.3 & 0 < t < 20 & \text{Parley's Walmart to Kimball} \\
56.3 & 20 < t < 32 & \text{Kimball to Wanship} \\
42.0 & 32 < t < 45 & \text{Wanship to Coalville} \\
48.9 & 45 < t < 52 & \text{Coalville to Echo Dam} \\
55.0 & 52 < t < 70 & \text{Echo Dam to 75 mph sign} \\
65.0 & 70 < t < 106 & \text{75 mph sign to Evanston} \\
68.7 & 106 < t < 375 & \text{Evanston to Laramie} \\
61.2 & 375 < t < 425 & \text{Laramie to Cheyenne} \\
69.7 & 425 < t < 462 & \text{Cheyenne to Pine bluffs} \\
0.00 & 462 < t < \infty & \text{SUV stopped}
\end{array}
\end{cases}
$$

The velocity function $V_{\text{pc}}(t)$ is piecewise continuous, because it has the general form

$$
f(t) = \begin{cases}
f_1(t) & t_1 < t < t_2 \\
f_2(t) & t_2 < t < t_3 \\
\vdots & \vdots \\
f_n(t) & t_n < t < t_{n+1}
\end{cases}
$$

where functions $f_1, f_2, \ldots, f_n$ are **continuous on the whole real line** $-\infty < t < \infty$. We don't define $f(t)$ at division points, because of many possible ways to make the definition. As long as these values are not used, then it will make no difference. Both right and left hand limits exist at a division point. For Laplace theory, we like the definition $f(t_k) = \lim_{h \to 0+} f(t_k + h)$, which makes the function right-continuous.

The Problem. The SUV travels from $t = 0$ to $t = \frac{462}{60} = 7.7$ hours. The odometer trip meter reading $x(t)$ is in miles (assume $x(0) = 0$). The function $V_{\text{pc}}(t)$ is an approximation to the speedometer reading. Laplace's method can solve the approximation model

$$
\frac{dx}{dt} = V_{\text{pc}}(60t), \quad x(0) = 0, \quad x \text{ in miles, } t \text{ in hours,}
$$

obtaining $x(t) = \int_0^t V_{\mathsf{pc}}(60w)dw$, the same result as the method of quadrature. Show the details. Then display the piecewise linear continuous trip meter reading $x(t)$.

Solution.

Method of Quadrature. The meaning of the differential equation is that $x'(t)$ is piecewise continuous. We want $x(t)$ to be continuous, because it is the odometer trip meter reading. But $x'(t)$ cannot be continuous, if we require $\frac{dx}{dt} = V_{\mathsf{pc}}(60t)$, because the right side is piecewise defined and discontinuous at division points.

Theorem (Fundamental Theorem of Calculus)
If $f'(x)$ is piecewise continuous and $f(x)$ is continuous on $a \le x \le b$, then $\int_a^b f'(x)dx = f(b) - f(a)$.

The theorem implies that the method of quadrature works for the equation $x'(t) = V_{\mathsf{pc}}(60t)$. The quadrature method gives the correct answer

$$x(t) = \int_0^t V_{\mathsf{pc}}(60w)dw.$$

Another plan is to split $x'(t) = V_{\mathsf{pc}}(60t)$ into 10 simple equations, $x' = 54.3$, $x(0) = 0$ on $0 \le t < 20$ being the first equation. The next equation is $x' = 56.3$, $x(20) = x_0$, on $20 < t < 32$. To make $x(t)$ continuous, we must choose $x_0 = 1086$, which is the value at the division point $t = 20$ assumed by the first problem ($x' = 54.3$, $x(0) = 0$ on $0 \le t < 20$). This tedious process has to be continued for all 10 segments. The result is that $x(t)$ is **piecewise linear** between division points.

Laplace's Method. The piecewise continuous input $V_{\mathsf{pc}}(60t)$ is of exponential order, because it is zero after $t = 462/60$. Laplace theory says it has a Laplace transform $\mathcal{L}(V_{\mathsf{pc}}(60t))$. Assuming a continuous solution $x(t)$, with $x'(t)$ piecewise continuous, then the equation to be satisfied is

$$s\mathcal{L}(x(t)) - x(0) = \mathcal{L}(x'(t)) = \mathcal{L}(V_{\mathsf{pc}}(60t)).$$

The **Laplace integral theorem** implies

$$\mathcal{L}(x(t)) = \frac{1}{s}\mathcal{L}(V_{\mathsf{pc}}(60t)) = \mathcal{L}\left(\int_0^t V_{\mathsf{pc}}(60w)dw\right).$$

Lerch's theorem then implies that the symbol $\mathcal{L}$ cancel from each side, giving the odometer trip meter reading in terms of the integral of the piecewise continuous input $V_{\mathsf{pc}}(60t)$:

$$x(t) = \int_0^t V_{\mathsf{pc}}(60w)dw.$$

We'll use technology to program and evaluate the integral, even though it can be done by hand. The plan is to plot the trip meter reading, then comment on the slow and fast segments of the route, by using a clever plot involving the average speed. The last display is the piecewise linear trip meter reading $x(t)$.

Maple

```
Xpc:=t->piecewise(t<0,0,
t < 20 ,54.3, t < 32, 56.3, t < 45, 42,  t < 52, 48.9,
t < 70 ,55, t < 106, 65, t < 375, 68.7, t < 425, 61.2,
t < 462, 69.7, 0.0);
X:=t->int(Xpc(60*w),w=0..t);
plot(X(t),t=0..480/60); # Almost a straight line.
```

Average Speed

Define the average value of a function $f(w)$ on $a \le w \le b$ by $\frac{1}{b-a}\int_a^b f(w)dw$. Then the average speed in the example is

$$\frac{\int_0^{462/60} V_{\mathsf{pc}}(60w)dw}{462/60} = 65.14956710.$$

A Clever Plot

An average driver would try to maintain 65.15 mph. The clever plot will create a graphic of $x(t)-65.15t$ on interval $0 \le t \le T_1$, where T_1 is the 471 mile trip time at 65.15 mph.

```
# Maple code
Xpc:=t->piecewise(t<0,0,
t < 20 ,54.3, t < 32, 56.3, t < 45, 42,  t < 52, 48.9,
t < 70 ,55, t < 106, 65, t < 375, 68.7, t < 425, 61.2,
t < 462, 69.7, 0.0);
X:=t->int(Xpc(60*w),w=0..t);
AVEspeed:=X(462/60)/(462/60); # AVEspeed = 65.14956710 mph
T1:=solve(AVEspeed*t=471,t);   # T1 = 7.229518491 hours
plot(X(t)-AVEspeed*t,t=0..T1);
```
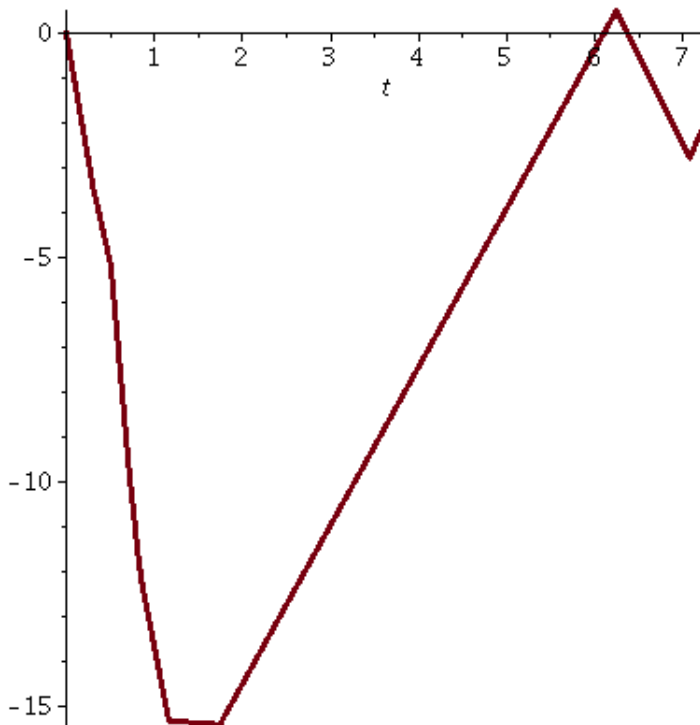

We see from the graphic that segments of the road cause a slowdown of up to 15 mph, but for a brief interval it is possible to exceed the average speed, due to a 75 mph speed limit.

```
# Maple code for piecewise linear display
X:=t->int(Xpc(60*w),w=0..t);
convert(X(t),piecewise,t):evalf(%,4);
```

$$
\text{Trip meter at time } t = \begin{cases}
0.0 & t \le 0.0 \\
54.30\,t & t \le 0.3333 \\
56.30\,t - 0.6667 & t \le 0.53 \\
42.0\,t + 6.960 & t \le 0.75 \\
48.90\,t + 1.785 & t \le 0.8667 \\
55.0\,t - 3.502 & t \le 1.167 \\
65.0\,t - 15.17 & t \le 1.767 \\
68.70\,t - 21.70 & t \le 6.25 \\
61.20\,t + 25.17 & t \le 7.083 \\
69.70\,t - 35.04 & t \le 7.7 \\
501.7 & 7.7 < t
\end{cases}
$$