Differential Equations 2280

Sample Midterm Exam 2 with Solutions
Exam Date: 1 April 2016 at 12:50pm

Instructions: This in-class exam is 50 minutes. No calculators, notes, tables or books. No answer check is expected. Details count 3/4, answers count 1/4. Problems below cover the possibilities, but the exam day content will be much less, as was the case for Exam 1.

1. **(Chapter 3)**
(a) [50%] Find by any applicable method the steady-state periodic solution for the current equation $I'' + 2I' + 5I = -10\sin(t)$.
(b) [50%] Find by variation of parameters a particular solution y_p for the equation $y'' = 1 - x$. Show all steps in variation of parameters. Check the answer by quadrature. This problem is moved to Exam 3.

Answer:
Part (a) Answer: $I_{SS}(t) = \cos t - 2\sin t$.
Variation of Parameters.
Solve $x'' + 2x' + 5x = 0$ to get $x_h = c_1x_1 + c_2x_2$, $x_1 = e^{-t}\cos 2t$, $x_2 = e^{-t}\sin 2t$. Compute the Wronskian $W = x_1x_2' - x_1'x_2 = 4e^{-2t}$. Then for $f(t) = -10\sin(t)$,

$$x_p = x_1\int x_2\frac{-f}{W}dt + x_2\int x_1\frac{f}{W}dt.$$

The integrations are too difficult, so the method won't be pursued.
Undetermined Coefficients.
The trial solution by Rule I is $I = d_1\cos t + d_2\sin t$. The homogeneous solutions have exponential factors, therefore the Euler solution atoms in the trial solution cannot be solutions of the homogeneous problem, hence Rule II does not apply.
Substitute the trial solution into the non-homogeneous equation to obtain the answers $d_1 = 1$, $d_2 = -2$. The unique periodic solution I_{SS} is extracted from the general solution $I = I_h + I_p$ by crossing out all negative exponential terms (terms which limit to zero at infinity). Because $I_p = d_1\cos t + d_2\sin t = \cos t - 2\sin t$ and the homogeneous solution x_h has negative exponential terms, then

$$I_{SS} = \cos t - 2\sin t.$$

Laplace Theory.
Plan: Find the general solution, then extract the steady-state solution by dropping negative exponential terms. The computation can use initial data $I(0) = I'(0) = 0$, because every particular solution contains the terms of the steady-state solution. Some details:
$(s^2 + 2s + 5)\mathcal{L}(I) = \dfrac{-10}{s^2+1}$
$\mathcal{L}(I) = \dfrac{-10}{(s^2+1)(s^2+2s+5)}$
$\mathcal{L}(I) = \dfrac{-10}{(s^2+1)((s+1)^2+4)}$
$\mathcal{L}(I) = \dfrac{s-2}{s^2+1} - \dfrac{s}{(s+1)^2+4}$
$\mathcal{L}(I) = \dfrac{s}{s^2+1} - 2\dfrac{1}{s^2+1} - \dfrac{s+1}{(s+1)^2+4} + \dfrac{1}{2}\dfrac{2}{(s+1)^2+4}$
$\mathcal{L}(I) = \mathcal{L}(\cos t) - 2\mathcal{L}(\sin t) - \mathcal{L}(e^{-t}\cos 2t) + \dfrac{1}{2}\mathcal{L}(e^{-t}\sin 2t)$
$I(t) = \cos t - 2\sin t - e^{-t}\cos 2t + \dfrac{1}{2}e^{-t}\sin 2t$, by Lerch's Theorem.

Dropping the negative exponential terms gives the steady-state solution $I_{SS}(t) = \cos t - 2\sin t$.

Part (b) Answer: $y_p = \dfrac{x^2}{2} - \dfrac{x^3}{6}$.

Variation of Parameters.

Solve $y'' = 0$ to get $y_h = c_1y_1 + c_2y_2$, $y_1 = 1$, $y_2 = x$. Compute the Wronskian $W = y_1y_2' - y_1'y_2 = 1$. Then for $f(t) = 1 - x$,

$y_p = y_1 \int y_2 \frac{-f}{W} dx + y_2 \int y_1 \frac{f}{W} dx$,

$y_p = 1 \int -x(1-x)dx + x \int 1(1-x)dx$,

$y_p = -1(x^2/2 - x^3/3) + x(x - x^2/2)$,

$y_p = x^2/2 - x^3/6$.

This answer is checked by quadrature, applied twice to $y'' = 1 - x$ with initial conditions zero.

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2. **(Chapters 1, 2, 3)**

(2a) [20%] Solve $2v'(t) = -8 + \dfrac{2}{2t+1}v(t)$, $v(0) = -4$. Show all integrating factor steps.

(2b) [10%] Solve for the general solution: $y'' + 4y' + 6y = 0$.

(2c) [10%] Solve for the general solution of the homogeneous constant-coefficient differential equation whose characteristic equation is $r(r^2 + r)^2(r^2 + 9)^2 = 0$.

(2d) [20%] Find a linear homogeneous constant coefficient differential equation of lowest order which has a particular solution $y = x + \sin\sqrt{2}x + e^{-x}\cos 3x$.

(2e) [15%] A particular solution of the equation $mx'' + cx' + kx = F_0\cos(2t)$ happens to be $x(t) = 11\cos 2t + e^{-t}\sin\sqrt{11}t - \sqrt{11}\sin 2t$. Assume m, c, k all positive. Find the unique periodic steady-state solution x_{SS}.

(2f) [25%] Determine for $y''' + y'' = 100x^2 + \sin x$ the shortest trial solution for y_p according to the method of undetermined coefficients. **Do not evaluate** the undetermined coefficients!

Answer:

(2a) $v(t) = -4 - 8t$

(2b) $r^2 + 4r + 6 = 0$, $y = c_1y_1 + c_2y_2$, $y_1 = e^{-2x}\cos\sqrt{2}x$, $y_2 = e^{-2x}\sin\sqrt{2}x$.

(2c) Write as $r^3(r+1)^2(r^2+9)^2 = 0$. Then y is a linear combination of the atoms 1, x, x^2, e^{-x}, xe^{-x}, $\cos 3x$, $x\cos 3x$, $\sin 3x$, $x\sin 3x$.

(2d) The atoms that appear in $y(x)$ are $x, \sin\sqrt{2}x, e^{-x}\cos 3x$. Derivatives of these atoms create a longer list: $1, x, \cos\sqrt{2}x, \sin\sqrt{2}x, e^{-x}\cos 3x, e^{-x}\sin 3x$. These atoms correspond to characteristic equation roots $0, 0$; $\sqrt{2}i$, $-\sqrt{2}i$, $-1+3i$, $-1-3i$. Then the characteristic equation has factors r, r; x^2+2; $((r+1)^2+9)$. The product of these factors is the correct characteristic equation, which corresponds to the differential equation of least order such that $y(x)$ is a solution. Report $r^6 + 2r^5 + 12r^4 + 4r^3 + 20r^2 = 0$ as the characteristic equation or $y^{(6)} + 2y^{(5)} + 12y^{(4)} + 4y''' + 20y'' = 0$ as the differential equation.

(2e) It has to equal the terms left over after striking out the transient terms, those terms with limit zero at infinity. Then $x_{SS}(t) = 11\cos 2t - \sqrt{11}\sin 2t$.

(2f) The homogeneous solution is a linear combination of the atoms 1, x, e^{-x} because the characteristic polynomial has roots 0, 0, -1.

Rule $\boxed{1}$ An initial trial solution y is constructed for atoms 1, x, x^2, $\cos x$, $\sin x$ giving 3 groups, each group having the same base atom:

$$\begin{aligned} y &= y_1 + y_2 + y_3, \\ y_1 &= d_1 + d_2x + d_3x^2, \\ y_2 &= d_4\cos x, \\ y_3 &= d_5\sin x. \end{aligned}$$

Linear combinations of the listed independent atoms are supposed to reproduce, by specialization of constants, all derivatives of the right side of the differential equation.

Rule $\boxed{2}$ The correction rule is applied individually to each of y_1, y_2, y_3.

Multiplication by x is done repeatedly, until the replacement atoms do not appear in atom list for the homogeneous differential equation. The result is the **shortest trial solution**

$$y = y_1 + y_2 + y_3 = (d_1 x^2 + d_2 x^3 + d_3 x^4) + (d_4 \cos x) + (d_5 \sin x).$$

Some facts: (1) If an Euler solution atom of the homogeneous equation appears in a group, then it is removed because of x-multiplication, but replaced by a new atom having the same base atom. (2) The number of terms in each of y_1 to y_3 is unchanged from Rule I to Rule II.

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3. **(Laplace Theory)**

(a) [50%] Solve by Laplace's method $x'' + 2x' + x = e^t$, $x(0) = x'(0) = 0$.

(b) [25%] Assume $f(t)$ is of exponential order. Find $f(t)$ in the relation

$$\left.\frac{d}{ds}\mathcal{L}(f(t))\right|_{s \to (s-3)} = \mathcal{L}(f(t) - t).$$

(c) [25%] Derive an integral formula for $y(t)$ by Laplace transform methods, explicitly using the Convolution Theorem, for the problem

$$y''(t) + 4y'(t) + 4y(t) = f(t), \quad y(0) = y'(0) = 0.$$

This is similar to a required homework problem from Chapter 7.

Answer:

(a)

$x(t) = -1/4\,e^{-t} - 1/2\,e^{-t} t + 1/4\,e^t$

An intermediate step is $\mathcal{L}(x(t)) = \dfrac{1}{(s-1)(s+1)^2}$. The solution uses partial fractions $\dfrac{1}{(s-1)(s+1)^2} = \dfrac{A}{s-1} + \dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$, with answers $A = 1/4$, $B = -1/4$, $C = -1/2$.

(b)

Replace by the shift theorem and the s-differentiation theorem the given equation by

$$\mathcal{L}\left((-t)f(t)e^{3t}\right) = \mathcal{L}(f(t) - t).$$

Then Lerch's theorem cancels $\mathcal{L}$ to give $-te^{3t}f(t) = f(t) - t$. Solve for

$$f(t) = \frac{t}{1 + te^{3t}}.$$

(c)

The main steps are:

$(s^2 + 4s + 4)\mathcal{L}(y(t)) = \mathcal{L}(f(t))$,

$\mathcal{L}(y(t)) = \dfrac{1}{(s+2)^2}\mathcal{L}(f(t))$,

$\mathcal{L}(y(t)) = \mathcal{L}(te^{-2t})\mathcal{L}(f(t))$, by the first shifting theorem,

$\mathcal{L}(y(t)) = \mathcal{L}(\text{convolution of } te^{-2t} \text{ and } f(t))$, by the Convolution Theorem,

$\mathcal{L}(y(t)) = \mathcal{L}\left(\int_0^t xe^{-2x}f(t-x)dx\right)$, insert definition of convolution,

$y(t) = \int_0^t xe^{-2x}f(t-x)dx$, by Lerch's Theorem.

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4. (Laplace Theory)

(4a) [20%] Explain Laplace's Method, as applied to the differential equation $x'(t)+2x(t) = e^t$, $x(0) = 1$. Reference only. Not to appear on any exam.

(4b) [15%] Solve $\mathcal{L}(f(t)) = \dfrac{100}{(s^2+1)(s^2+4)}$ for $f(t)$. [Edited 29 March]

(4c) [15%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{1}{s^2(s+3)}$.

(4d) [10%] Find $\mathcal{L}(f)$ given $f(t) = (-t)e^{2t}\sin(3t)$.

(4e) [20%] Solve $x''' + x'' = 0$, $x(0) = 1$, $x'(0) = 0$, $x''(0) = 0$ by Laplace's Method.

(4f) [20%] Solve the system $x' = x + y$, $y' = x - y + 2$, $x(0) = 0$, $y(0) = 0$ by Laplace's Method.

Answer:

(4a) Laplace's method explained.

The first step transforms the equation using the parts formula and initial data to get

$$(s+2)\mathcal{L}(x) = 1 + \mathcal{L}(e^t).$$

The forward Laplace table applies to evaluate $\mathcal{L}(e^t)$. Then write, after a division, the isolated formula for $\mathcal{L}(x)$:

$$\mathcal{L}(x) = \frac{1 + 1/(s-1)}{s+2} = \frac{s}{(s-1)(s+2)}.$$

Partial fraction methods plus the backward Laplace table imply

$$\mathcal{L}(x) = \frac{a}{s-1} + \frac{b}{s+2} = \mathcal{L}(ae^t + be^{-2t})$$

and then $x(t) = ae^t + be^{-2t}$ by Lerch's theorem. The constants are $a = 1/3$, $b = 2/3$.

(4b) $\mathcal{L}(f) = \frac{100}{(u+1)(u+4)} = \frac{100/3}{u+1} + \frac{-100/3}{u+4}$ where $u = s^2$. Then $\mathcal{L}(f) = \frac{100}{3}(\frac{1}{s^2+1} - \frac{1}{s^2+4}) = \frac{100}{3}\mathcal{L}(\sin t - \frac{1}{2}\sin 2t)$ implies $f(t) = \frac{100}{3}(\sin t - \frac{1}{2}\sin 2t)$.

(4c) $\mathcal{L}(f) = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+3} = \mathcal{L}(a + bt + ce^{-3t})$ implies $f(t) = a + bt + ce^{-3t}$. The constants, by Heaviside coverup, are $a = -1/9$, $b = 1/3$, $c = 1/9$.

(4d) $\mathcal{L}(f) = \frac{d}{ds}\mathcal{L}(e^{2t}\sin 3t)$ by the s-differentiation theorem. The first shifting theorem implies $\mathcal{L}(e^{2t}\sin 3t) = \mathcal{L}(\sin 3t)|_{s\to(s-2)}$. Finally, the forward table implies $\mathcal{L}(f) = \frac{d}{ds}\left(\frac{1}{(s-2)^2+9}\right) = \frac{-2(s-2)}{((s-2)^2+9)^2}$.

(4e) The answer is $x(t) = 1$, by guessing, then checking the answer. The Laplace details jump through hoops to arrive at $(s^3+s^2)\mathcal{L}(x(t)) = s^2 + s$, or simply $\mathcal{L}(x(t)) = 1/s$. Then $x(t) = 1$.

(4f) The transformed system is

$$\begin{aligned} (s-1)\mathcal{L}(x) &+ (-1)\mathcal{L}(y) &= 0, \\ (-1)\mathcal{L}(x) &+ (s+1)\mathcal{L}(y) &= \mathcal{L}(2). \end{aligned}$$

Then $\mathcal{L}(2) = 2/s$ and Cramer's Rule gives the formulas

$$\mathcal{L}(x) = \frac{2}{s(s^2-2)}, \quad \mathcal{L}(y) = \frac{2(s-1)}{s(s^2-2)}.$$

After partial fractions and the backward table,

$$x = -1 + \cosh(\sqrt{2}t), \quad y = \sqrt{2}\sinh(\sqrt{2}t) - \cosh(\sqrt{2}t) + 1.$$

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5. (Laplace Theory)

(a) [30%] Solve $\mathcal{L}(f(t)) = \dfrac{1}{(s^2+s)(s^2-s)}$ for $f(t)$.

(b) [20%] Solve for $f(t)$ in the equation $\mathcal{L}(f(t)) = \dfrac{s+1}{s^2+4s+5}$.

(c) [20%] Let $u(t)$ denote the unit step. Solve for $f(t)$ in the relation

$$\mathcal{L}(f(t)) = \frac{d}{ds}\mathcal{L}(u(t-1)\sin 2t)$$

Remark: This is not a second shifting theorem problem.

(d) [30%] Compute $\mathcal{L}(e^{2t}f(t))$ for

$$f(t) = \frac{e^t - e^{-t}}{t}.$$

Answer:

(a) $f(t) = \sinh(t) - t = \frac{1}{2}e^t - \frac{1}{2}e^{-t} - t$

(b) $f(t) = e^{-2t}(\cos(t) - \sin(t))$

(c) Replace d/ds by factor $(-t)$ in the Laplace integrand:

$$\mathcal{L}(f(t)) = \mathcal{L}((-t)\sin(2t)u(t-1))$$

Apply Lerch's theorem to cancel $\mathcal{L}$ on each side, obtaining the answer

$$f(t)) = (-t)\sin(2t)u(t-1).$$

(d) The first shifting theorem reduces the problem to computing $\mathcal{L}(f(t))$.

$\mathcal{L}(tf(t)) = \mathcal{L}(e^t - e^{-t}) = \dfrac{1}{s-1} - \dfrac{1}{s+1}$

$-\dfrac{d}{ds}\mathcal{L}(f(t)) = \dfrac{1}{s-1} - \dfrac{1}{s+1}$, by the s-differentiation theorem,

Then $F(s) = \mathcal{L}(f(t))$ satisfies a first order quadrature equation $F'(s) = h(s)$ with solution $F(s) = \ln|s+1| - \ln|s-1| + c = \ln\left|\frac{s+1}{s-1}\right| + c$ for some constant c. Because $F = 0$ at $s = \infty$ (a basic theorem for functions of exponential order) and $\ln|1| = 0$, then $c = 0$ and $\mathcal{L}(f(t)) = F(s) = \ln|s+1| - \ln|s-1| = \ln\left|\frac{s+1}{s-1}\right|$. Then the shifting theorem implies

$$\mathcal{L}\left(e^{2t}f(t)\right) = \mathcal{L}(f(t))|_{s:=s-2} = \ln\left|\frac{s-1}{s-3}\right|.$$

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6. (Systems of Differential Equations)

The eigenanalysis method says that, for a 3×3 system $\mathbf{x}' = A\mathbf{x}$, the general solution is $\mathbf{x}(t) = c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + c_3\mathbf{v}_3e^{\lambda_3 t}$. In the solution formula, $(\lambda_i, \mathbf{v}_i)$, $i = 1, 2, 3$, is an eigenpair of A. Given

$$A = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix},$$

then

(a) [75%] Display eigenanalysis details for A.

(b) [25%] Display the solution $\mathbf{x}(t)$ of $\mathbf{x}'(t) = A\mathbf{x}(t)$.

(c) Repeat (a), (b) for the matrix $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 0 & 0 & 7 \end{bmatrix}$.

Answer:

(a): The details should solve the equation $|A - \lambda I| = 0$ for three values $\lambda = 5, 4, 3$. Then solve the three systems $(A - \lambda I)\vec{v} = \vec{0}$ for eigenvector $\vec{v}$, for $\lambda = 5, 4, 3$.

The eigenpairs are

$$5, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 4, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad 3, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(b): The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^{3t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

(c): The eigenpairs are

$$6, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}; \quad 7, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \quad 4, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

and The eigenanalysis method implies

$$\mathbf{x}(t) = c_1 e^{6t} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c_2 e^{7t} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

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7. **(Systems of Differential Equations)**

(a) [30%] Find the eigenvalues of the matrix $A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}$.

(b) [20%] Justify that eigenvectors of A corresponding to the eigenvalues in increasing order are the four vectors

$$\begin{pmatrix} 1 \\ -5 \\ -3 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

(c) [50%] Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Eigenanalysis method. This is identical to the answer

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Answer:

(a) Eigenvalues are $\lambda = 2, 3, 4, 5$.
Define

$$A = \begin{bmatrix} 4 & 1 & -1 & 0 \\ 1 & 4 & -2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 4 \end{bmatrix}.$$

Subtract λ from the diagonal elements of A and expand the determinant $\det(A - \lambda I)$ to obtain the characteristic polynomial $(2-\lambda)(3-\lambda)(4-\lambda)(5-\lambda) = 0$. The eigenvalues are the roots: $\lambda = 2, 3, 4, 5$. Used here was the *cofactor rule* for determinants. Also possible is the special result for block matrices, $\begin{vmatrix} B_1 & 0 \\ C & B_2 \end{vmatrix} = |B_1||B_2|$. Sarrus' rule does not apply for 4×4 determinants (an error) and the triangular rule likewise does not directly apply (another error).
(b) To be justified is $AP = PD$ where $D = \mathbf{diag}(2, 3, 4, 5)$ is the diagonal matrix of eigenvalues (see part (a)) and P is the augmented matrix of eigenvectors supplied above. Matrix multiply can check the answer, by expanding each side of $AP = PD$.
Alternative method:
Solve $(A - \lambda I)\vec{v} = \vec{0}$ four times, for $\lambda = 2, 3, 4, 5$. Each is a homogeneous system of linear algebraic equations, reduced to RREF by swap, combo, multiply. Each eigenvector answer is Strang's Special Solution.
(c) Because the eigenvalues are $\lambda = 2, 3, 4, 5$, then the solution is a vector linear combination of the Euler solution atoms $e^{2t}, e^{3t}, e^{4t}, e^{5t}$:

$$\mathbf{u}(t) = \vec{d}_1 e^{2t} + \vec{d}_2 e^{3t} + \vec{d}_3 e^{4t} + \vec{d}_4 e^{5t}.$$

According to the theory, $\vec{d}_j = c_j \vec{v}_j$, where $(\lambda_1, \vec{v}_1)$, ..., $(\lambda_4, \vec{v}_4)$ are the eigenpairs of A and c_1, c_2, c_3, c_4 are invented symbols representing real, arbitrary constants. Then

$$\vec{u} = c_1 e^{2t} \begin{pmatrix} 1 \\ -5 \\ -3 \\ 3 \end{pmatrix} + c_2 e^{3t} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_3 e^{4t} \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + c_4 e^{5t} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

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8. (Systems of Differential Equations)

(a) [100%] The eigenvalues are 3, 5 for the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}$.

Display the general solution of $\mathbf{u}' = A\mathbf{u}$ according to the Cayley-Hamilton-Ziebur shortcut (textbook chapters 4,5). Assume initial condition $\vec{u}_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Answer:
(a) **Cayley-Hamilton Ziebur Shortcut**. The method says that the components $x(t), y(t)$ of the solution to the system

$$\vec{u}' = A\vec{u}, \quad \vec{u}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ are linear combinations of the Euler atoms found from the roots of the characteristic equation $|A - rI| = 0$. The roots are $r = 3, 5$ and the atoms are e^{3t}, e^{5t}. The scalar system is

$$\begin{cases} x'(t) &= 4x(t) + y(t), \\ y'(t) &= x(t) + 4y(t), \\ x(0) &= 1, \\ y(0) &= -1. \end{cases}$$

The C-H-Z method implies $x(t) = c_1 e^{3t} + c_2 e^{5t}$, but c_1, c_2 are not arbitrary constants: they are determined by the initial conditions $x(0) = 1, y(0) = -1$. Then $x' = 4x + y$ can be solved for y to obtain $y(t) = x'(t) - 4x(t)$. Substitute expression $x(t) = c_1 e^{3t} + c_2 e^{5t}$ into $y(t) = x'(t) - 4x(t)$ to obtain

$$y(t) = x'(t) - 4x(t) = 3c_1 e^{3t} + 5c_2 e^{5t} - 4(c_1 e^{3t} + c_2 e^{5t}) = -c_1 e^{3t} + c_2 e^{5t}.$$

Then

$$\begin{cases} x(t) &= c_1 e^{3t} + c_2 e^{5t}, \\ y(t) &= -c_1 e^{3t} + c_2 e^{5t}. \end{cases} \tag{1}$$

Initial data $x(0) = 1, y(0) = -1$ are used in the last step, to evaluate c_1, c_2. Inserting these conditions produces a 2×2 linear system for c_1, c_2

$$\begin{cases} 0 &= c_1 e^0 + c_2 e^0, \\ 0 &= -c_1 e^0 + c_2 e^0. \end{cases}$$

The solution is $c_1 = 1$ and $c_2 = 0$, which implies the final answer $x(t) = e^{3t}$, $y(t) = -e^{3t}$.

Remark on Fundamental Matrices. The answer prior to evaluation of c_1, c_2 can be written as

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{5t} \\ -e^{3t} & e^{5t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

The matrix $\Phi(t) = \begin{pmatrix} e^{3t} & e^{5t} \\ -e^{3t} & e^{5t} \end{pmatrix}$ is called a **fundamental matrix**, because it is nonsingular and satisfies $\Phi' = A\Phi$ (its columns are solutions of $\vec{u}' = A\vec{u}$). In terms of Φ,

$$e^{At} = \Phi(t)\Phi^{-1}(0).$$

This formula gives an alternative way to compute e^{At} by using the Cayley-Hamilton-Ziebur shortcut. Please observe that the columns of Φ are the formal partial derivatives of the vector solution $\vec{u}$ on the symbols c_1, c_2. Partial derivatives on symbols is a general method for discovering basis vectors. Therefore, Φ can be written directly from equations (1).
