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Math 2270

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Fractals in Nature

Introduction:

Fractals are well known in popular culture; however, many people do not understand their mathematical foundation or definition. As varied as fractals can be, they all contain a certain set of characteristics. Fractals are said to be self-similar, meaning that a pattern is followed for any scale of the fractal. This can be an exact self-similarity, as with the Sierpinski Triangle, or even random self-similarity as can be found in nature. They are also potentially infinitely complex, stemming from the fact that they can be formed by iterated function systems. Such fractals, when an area is magnified, reveal a similar fractal with equal beauty. Thus, fractals catch the heart of mathematicians and non-mathematicians alike, appearing not only in the theoretical world mathematics, but also in nature.

Iterated Function Systems:

Iterated function systems are simply the result of a function applied repeatedly to a certain x in its domain. In the case of fractals, x is a shape that is repeatedly transformed by the function. The fractal's requirement for self-similarity is met through the copied shape that is then put through a transformation for each iteration of the function. This transformation results in a contraction of the shape, which means the shape becomes smaller as the number of iterations increases. Additionally, the function usually moves the shape to a new location.

For each iteration, the function is applied to each new shape created by the iteration coming directly before. This sequence then creates the orbit of x, defined as:

$$F(x),\ F(F(x)) = F^2(x), \dots F^n(x)$$

As n goes to infinity, this sequence creates the self-similarity of fractals and, in a contraction mapping, it means that as we zoom in on one a point in the fractal, we'll see a similar fractal repeated there, in infinite complexity.

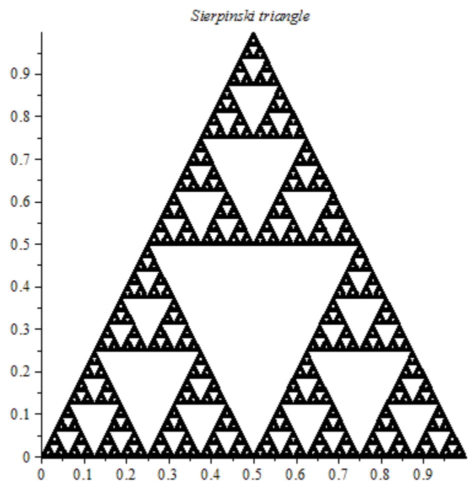
This mapping can be defined by:

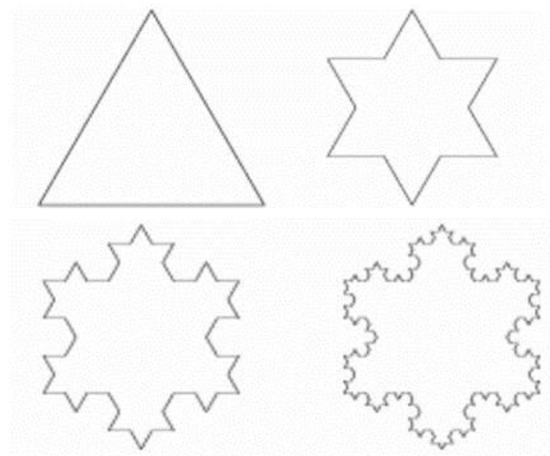
$$F_1:\ x \rightarrow A_1x + b_1,\ F_2:\ x \rightarrow A_2x + b_2,\ \dots\ F_n:\ x \rightarrow A_nx + b_n$$

where A is a rotation and scalar matrix and x and b are vectors. The matrix multiply performs the transformation while adding b allows the vector x to be adjusted in its location, not just contracted or rotated. Each F represents a unique copy of the current set. After the iteration, these copies are merged to form a new set which is then put through all the transformations during the next iteration. Because this represents a contraction mapping, the set will converge to form an attractor which is the fractal (Walsh, 1996).

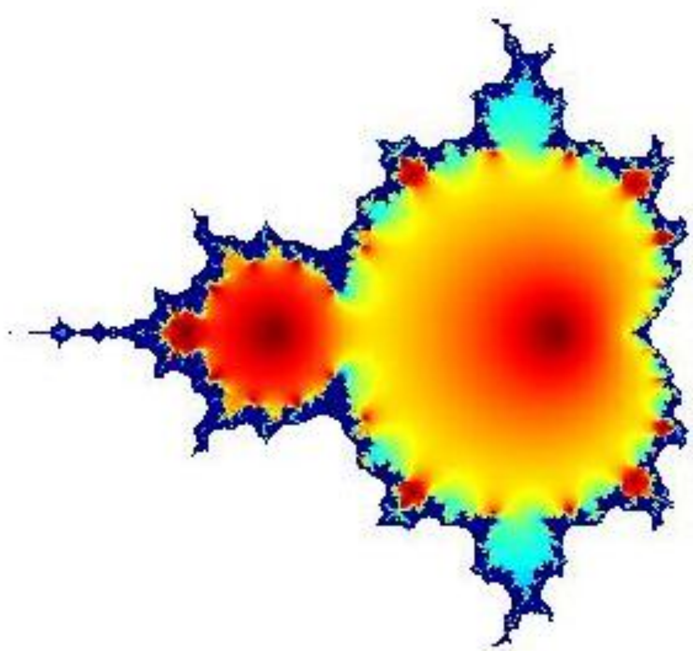
Famous Fractals:

One of the most famous fractals is the Sierpinski Triangle (shown below). It was created by Waclaw Sierpinski by taking a solid, point-up, equilateral triangle and removing a point-down equilateral triangle with vertices on the midpoints of the original triangle. For the resulting three solid, point-up smaller triangles, again remove a point-down triangle. Then repeat indefinitely for every solid, point-up triangle. Thus each triangle becomes a miniature copy of the previous triangle, which is how it is self-similar (Korevaar) . As the iterations could be infinite, the complexity is potentially infinite. The Koch snowflake follows a similar pattern, but with triangles being added to the sides of triangles as can be seen below.





The Mandelbrot Set is an incredibly well known escape-time fractal. It is generated by iterating $z_{n+1} = z_n^2 + c$ where c is a complex number and the shape is based off of where z is bounded. Ultimately it looks like a cardioid with growths coming off of it. If one of the growths is magnified, it will display quasi self-similarity. This means that it will look similar to the original Mandelbrot cardioid with growths, but is not exactly similar. The creation of the Mandelbrot set is again potentially infinitely complex (Šupina, 2006).



```matlab
% Matlab code to generate the above fractal
clear;
x=linspace(-.6-1.5,-.6+1.5,400);
y=linspace(-1.5,1.5,400);
[X,Y]=meshgrid(x,y);
Z=zeros(400);
C=X+1j*Y;
for k=1:20;
  Z=Z.^2+C;
  W=exp(-abs(Z));
end
colormap( jet(256));
pcolor(W);
shading flat;
axis('square','equal','off');
```

Natural Fractals:

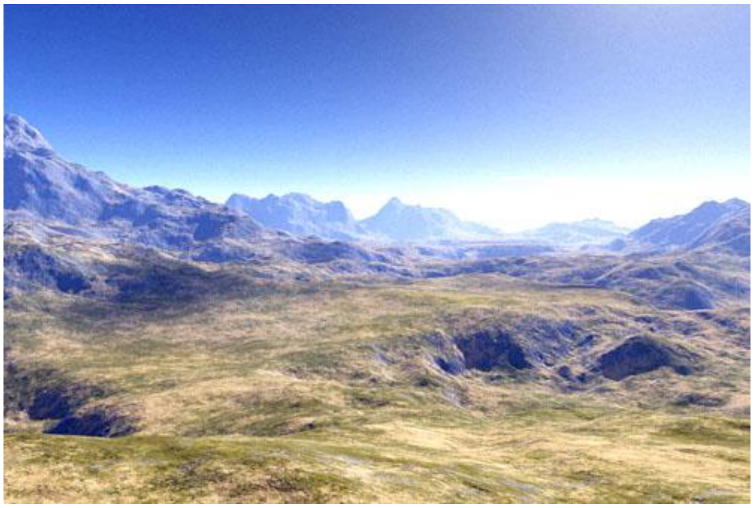
Naturally occurring fractals are typically referred to as approximate fractals, since they don't exhibit exact self-similarity, but have self-similar aspects to their patterns. Also, natural fractals occur over finite scale ranges, while mathematical fractals are theoretically infinite. (Oddee, 2008)

Approximate fractals can be found everywhere, from landforms to weather and even living creatures. When viewed from above, mountain ranges have a very noticeable self-similar pattern, as well as canyons formed by countless years of erosion. River networks also show fractal patterns as tributaries and streams branch off from rivers. Lightning, clouds, and snowflakes exhibit self-similar patterns, too. Fractals can also be found on living creatures, such as shells, animal coloration patterns, blood vessels, and plants such as trees, leaves, ferns, and even Romanesco broccoli, as seen in the following picture. (McNally, 2010)

Interestingly enough, many naturally occurring fractals have similar geometric patterns, although created by vastly different means. For example, the fractal pattern created by lightning, blood vessels, river networks, trees, and many other natural phenomena all have similar branching aspects. This shows that each natural phenomenon's pattern isn't entirely unique; many patterns found across nature are similar to each other as well as being self-similar. These natural fractal patterns are simple and efficient, and since nature favors simplicity and efficiency, the same distinctive patterns are seen throughout nature. (Miqel, 2007)

Remarkably, many of these naturally occurring fractal patterns can be modeled mathematically. As a typical example, ferns follow a fractal pattern that has been modeled mathematically. Snowflakes have also been modeled by fractals, such as the famous Koch snowflake fractal mentioned earlier. More advanced natural occurrences of fractals have also been generated by computers using mathematical, iterated methods (Miqel, 2007). The following picture is an example of a digital landscape generated entirely by fractals.



This mathematical modeling of nature shows that fractals are very applicable outside of theoretical mathematics. Fractals aren’t just intriguing mathematical patterns; they are found virtually everywhere and help us make sense of the patterns of nature.

References

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