Week 5 Examples

Example 1: Verify writing matrix product $A\vec{x}$ as a linear combination of the columns of A, by expansion of both sides, then compare the sides for equality of vectors.

$$\begin{pmatrix} 7 & 5 \\ -2 & 8 \end{pmatrix}\begin{pmatrix} 3 \\ 9 \end{pmatrix} = 3\begin{pmatrix} 7 \\ -2 \end{pmatrix} + 9\begin{pmatrix} 5 \\ 8 \end{pmatrix} \quad \text{(i.e., } A\vec{x} = x_1(\text{Col 1 of } A) + x_2(\text{Col 2 of } A)\text{)}$$

Example 2: Display the 3×3 elementary matrix E for the given Toolkit operation.

(2a) Multiply row 2 of I by m [`mult(2,m)`] **(2b)** Swap rows 1 and 3 of I [`swap(1,3)`]
(2c) Add m times rows 1 of I to row 3 [`combo(1,3,m)`]

Example 3: Given elementary matrix E, find the unique Toolkit operation that created E from the identity I.

(3a) $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & m \\ 0 & 0 & 1 \end{pmatrix}$ **(3b)** $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **(3c)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & m \end{pmatrix}$ **(3d)** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & 0 & 1 \end{pmatrix}$ **(3e)** $\begin{pmatrix} m & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Answers: `combo(3,2,m), swap(1,2), mult(3,m), combo(1,3,m), mult(1,m)`

Example 4: Label the matrices of the previous example when $m = 1$ as $E_1, \ldots, E_5$. Find the product $A = E_1E_2E_3E_4E_5$ without using matrix multiply. **Answer**: $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, because $E_5 = I$, then apply Toolkit operations `combo(1,3,1), mult(3,1), swap(1,2), combo(3,2,1).`

Example 5: Find the inverses of the elementary matrices in Example 3.

Answer: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -m \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{m} \end{pmatrix}$ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & 0 & 1 \end{pmatrix}$ $\begin{pmatrix} \frac{1}{m} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Example 6: Find the inverse by methods 1 and 2 below: $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$ $\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$

Method 1. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $\Delta = ad - bc$.

Method 2. Toolkit reduction of $C =< A|I >$ to $< I|B >$ where $B = A^{-1}$.

Answers: $\begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$ $\begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{4} & \frac{1}{2} \end{pmatrix}$ $\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $\begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix}$

Example 7: Solve the equations $\begin{cases} -x_1 + x_2 = 4, \\ 2x_1 - x_2 = 1, \end{cases}$ by converting to a matrix equation $A\vec{x} = \vec{b}$ and solving for $\vec{x} = A^{-1}\vec{b}$. Check the answer.

Example 8: Which of the following matrices have inverses? Answer without finding the inverse.

(8a) $\begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$ **(8b)** $\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ **(8c)** $\begin{pmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ **(8d)** $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ **(8e)** $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Example 9: Find the vector general solution in the form $\vec{x} =$ homogeneous + particular.

(9a) $\begin{cases} -x_1 + x_2 = 4, \\ 2x_1 - x_2 = 1, \end{cases}$ **(9b)** $\begin{cases} -x_1 + x_2 + x_3 = 4, \\ \quad - 5x_2 - 3x_3 = -15, \\ -3x_1 - 2x_2 = -3. \end{cases}$ **(9c)** $\begin{cases} -x_1 + x_2 + 2x_3 = 4, \\ 2x_1 - x_2 = 1, \\ 0 = 0. \end{cases}$

Example 10: Evaluate determinants by Sarrus' 2×2 and 3×3 Rule.
Notation: Only determinants have vertical bars.

(10a) $\begin{vmatrix} -1 & 1 \\ 2 & -3 \end{vmatrix}$ **(10b)** $\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$ **(10c)** $\begin{vmatrix} -1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{vmatrix}$ **(10d)** $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix}$ **(10e)** $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$