Welding Torch Problem
Nyquist-Shannon Sampling Theorem

- Welding Torch Problem

    Model

    Solution

    Example 1

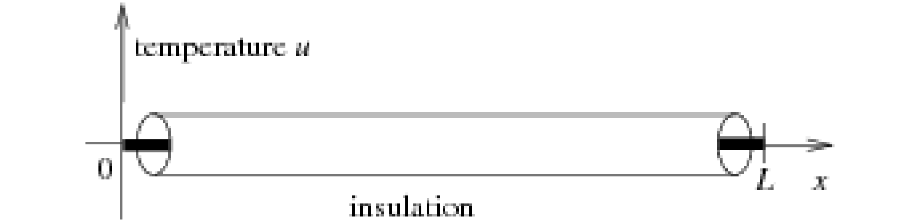
    Example 2

- Nyquist-Shannon Sampling Theorem

    Statement

    Whittaker-Shannon Interpolation Formula

Welding Torch Problem



Consider a long welding rod insulated laterally by a sheath. At position $x = 0$ a small hole is drilled into the sheath, then a torch injects energy into the hole, which spreads into the rod. The hole is closed, and we call this time $t = 0$. The problem is to determine the temperature $u(x, t)$ at location x along the rod and time $t > 0$.

Modeling.

$$\begin{aligned} u_t &= c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= f(x), \quad -\infty < x < \infty, \\ f(x) &= \delta(x) \quad \text{(Dirac impulse)} \end{aligned}$$

Solving the Welding Torch Problem

We will use the Heat Kernel to write the answer as

$$
\begin{aligned}
u(x,t) &= g_t * f \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} g_t(x-s)\delta(s)ds \\
&= \frac{1}{2c\sqrt{\pi t}}\, e^{-x^2/(4c^2 t)}
\end{aligned}
$$

The solution $u(x,t)$ can be checked to work in the PDE by direct differentiation. The mystery remaining is how to interpret the boundary condition $u(x,0) = \delta(x)$. This turns out to be an adventure into the **theory of distributions** (section 7.8, Asmar). The answer obtained is called a **weak solution** because of this technical difficulty.

Example 1. Cutting torch held for all time $t > 0$.

The physical model changes: the torch is applied at $x = 0$ for all time, and we never remove the torch or cover the hole drilled in the sheath. In addition, we assume the temperature at $t = 0$ is zero. We are adding energy constantly, so it is expected that the temperature $u(x,t)$ approaches infinity as t approaches infinity.

$$\begin{aligned} u_t &= \frac{1}{4}u_{xx} + \delta(x), \quad -\infty < x < \infty, \quad t > 0 \\ u(x,0) &= 0, \quad -\infty < x < \infty \end{aligned}$$

$$u(x,t) = \frac{2\sqrt{t}}{\sqrt{\pi}}\,e^{-x^2/t} - \frac{2|x|}{\sqrt{\pi}}\,\Gamma(0.5, x^2/t)$$

The **incomplete Gamma function** is defined by $\Gamma(a,x) = \int_x^\infty e^{-t}t^{a-1}\,dt$.

Because the Γ term is not positive, then $0 \le u(x,t) \le 2\sqrt{\frac{t}{\pi}}\,e^{-x^2/t}$. A limit at $x = 0$ gives $u(0,t) = 2\sqrt{t/\pi}$, meaning the temperature at $x = 0$ blows up like $\sqrt{t}$.

Example 2. Cutting torch held for 1 second.

The physical model: the torch is applied at $x = 0$ for one second and then we remove the torch and cover the hole that was drilled in the sheath. In addition, we assume the temperature at $t = 0$ is zero. We are adding energy only briefly, so it is expected that the temperature $u(x, t)$ is bounded.

$$\begin{aligned} u_t &= \frac{1}{4}u_{xx} + \delta(x)\,\mathrm{pulse}(t, 0, 1), \quad -\infty < x < \infty, \quad t > 0 \\ u(x, 0) &= 0, \quad -\infty < x < \infty \end{aligned}$$

The solution $u(x, t)$ has to agree with the solution $u_1(x, t)$ of the previous example until time $t = 1$. After this time, the temperature is $u(x, t) = u_1(x, t) - u_1(x, t - 1)$ (a calculation is required to see this result). Then

$$u(x, t) = \begin{cases} u_1(x, t) & 0 < t < 1, \\ u_1(x, t) - u_1(x, t - 1) & t > 1 \end{cases}$$

Nyquist-Shannon Sampling Theorem.

THEOREM. If a signal $f(t)$ contains no frequencies higher than W hertz, then the signal is completely determined from values $f(t_i)$ sampled at uniform spacing $\Delta t_i = t_i - t_{i-1}$ less than $\frac{1}{2W}$.

Bandlimited signals are perfectly reconstructed from infinitely many samples provided the bandwidth W is not greater than half the sampling rate (means $\Delta t < \frac{1}{2W}$).

Whittaker-Shannon Interpolation Formula

The formula uses the function $\textbf{sinc}(u) = \frac{\sin(u)}{u}$.

$$f(t) = \sum_{n=-\infty}^{\infty} f(nT)\,\textbf{sinc}\left(\pi \frac{t - nT}{T}\right)$$

Original Whittaker-Shannon Interpolation Formula

The formula uses the function $\mathbf{sinc}(u) = \frac{\sin(u)}{u}$. The original form of the formula is in terms of bandwidth W:

$$f(t) = \sum_{n=-\infty}^{\infty} f\left(\frac{n}{2W}\right) \mathbf{sinc}\left(\pi(2Wt - n)\right)$$