7.4 Cauchy-Euler Equation

The differential equation

$$a_n x^n y^{(n)} + a_{n-1} x^{n-1} y^{(n-1)} + \cdots + a_0 y = 0$$

is called the **Cauchy-Euler** differential equation of order n. The symbols a_i, $i = 0, \ldots, n$ are constants and $a_n \neq 0$.

The Cauchy-Euler equation is important in the theory of linear differential equations because it has direct application to **Fourier's method** in the study of partial differential equations. In particular, the second order Cauchy-Euler equation

$$ax^2y'' + bxy' + cy = 0$$

accounts for almost all such applications in applied literature.

A second argument for studying the Cauchy-Euler equation is theoretical: it is a single example of a differential equation with non-constant coefficients that has a known closed-form solution. This fact is due to a change of variables $(x, y) \longrightarrow (t, z)$ given by equations

$$x = e^t, \quad z(t) = y(x),$$

which changes the Cauchy-Euler equation into a constant-coefficient differential equation. Since the constant-coefficient equations have closed-form solutions, so also do the Cauchy-Euler equations.

Theorem 5 (Cauchy-Euler Equation)
The change of variables $x = e^t$, $z(t) = y(e^t)$ transforms the Cauchy-Euler equation

$$ax^2y'' + bxy' + cy = 0$$

into its equivalent constant-coefficient equation

$$a\frac{d}{dt}\left(\frac{d}{dt} - 1\right)z + b\frac{d}{dt}z + cz = 0.$$

The result is memorized by the general differentiation formula

$$x^k y^{(k)}(x) = \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - k + 1\right)z(t). \tag{1}$$

Proof: The equivalence is obtained from the formulas

$$y(x) = z(t), \quad xy'(x) = \frac{d}{dt}z(t), \quad x^2y''(x) = \frac{d}{dt}\left(\frac{d}{dt} - 1\right)z(t)$$

by direct replacement of terms in $ax^2y'' + bxy' + cy = 0$. It remains to establish the general identity (1), from which the replacements arise.

The method of proof is mathematical induction. The induction step uses the chain rule of calculus, which says that for $y = y(x)$ and $x = x(t)$,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

The identity (1) reduces to $y(x) = z(t)$ for $k = 0$. Assume it holds for a certain integer k; we prove it holds for $k + 1$, completing the induction.

Let us invoke the induction hypothesis LHS = RHS in (1) to write

$\frac{d}{dt}\mathsf{RHS} = \frac{d}{dt}\mathsf{LHS}$	Reverse sides.
$= \frac{dx}{dt}\frac{d}{dx}\mathsf{LHS}$	Apply the chain rule.
$= e^t\frac{d}{dx}\mathsf{LHS}$	Use $x = e^t$, $dx/dt = e^t$.
$= x\frac{d}{dx}\mathsf{LHS}$	Use $e^t = x$.
$= x\left(x^k y^{(k)}(x)\right)'$	Expand with $' = d/dx$.
$= x\left(kx^{k-1}y^{(k)}(x) + x^k y^{(k+1)}(x)\right)$	Apply the product rule.
$= k\,\mathsf{LHS} + x^{k+1}y^{(k+1)}(x)$	Use $x^k y^{(k)}(x) = \mathsf{LHS}$.
$= k\,\mathsf{RHS} + x^{k+1}y^{(k+1)}(x)$	Use hypothesis LHS = RHS.

Solve the resulting equation for $x^{k+1}y^{(k+1)}$. The result completes the induction. The details, which prove that (1) holds with k replaced by $k + 1$:

$$\begin{aligned} x^{k+1}y^{(k+1)} &= \frac{d}{dt}\mathsf{RHS} - k\,\mathsf{RHS} \\ &= \left(\frac{d}{dt} - k\right)\mathsf{RHS} \\ &= \left(\frac{d}{dt} - k\right)\frac{d}{dt}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - k + 1\right)z(t) \\ &= \frac{d}{dt}\left(\frac{d}{dt} - 1\right)\cdots\left(\frac{d}{dt} - k\right)z(t) \end{aligned}$$

1 **Example (How to Solve a Cauchy-Euler Equation)** Show the solution details for the equation

$$2x^2y'' + 4xy' + 3y = 0,$$

verifying general solution

$$y(x) = c_1x^{-1/2}\cos\left(\frac{\sqrt{5}}{2}\ln|x|\right) + c_2e^{-t/2}\sin\left(\frac{\sqrt{5}}{2}\ln|x|\right).$$

Solution: The characteristic equation $2r(r-1) + 4r + 3 = 0$ can be obtained as follows:

$2x^2y'' + 4xy' + 3y = 0$	Given differential equation.

$2x^2r(r-1)x^{r-2} + 4xrx^{r-1} + 3x^r = 0$	Use **Euler's substitution** $y = x^r$.
$2r(r-1) + 4r + 3 = 0$	Cancel x^r. **Characteristic equation** found.
$2r^2 + 2r + 3 = 0$	Standard quadratic equation.
$r = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}i$	Quadratic formula complex roots.

Cauchy-Euler Substitution. The second step is to use $y(x) = z(t)$ and $x = e^t$ to transform the differential equation. By Theorem 5,

$$2(d/dt)^2 z + 2(d/dt)z + 3z = 0,$$

a constant-coefficient equation. Because the roots of the characteristic equation $2r^2 + 2r + 3 = 0$ are $r = -1/2 \pm \sqrt{5}i/2$, then the Euler solution atoms are

$$e^{-t/2} \cos\left(\frac{\sqrt{5}}{2}t\right), \quad e^{-t/2} \sin\left(\frac{\sqrt{5}}{2}t\right).$$

Back-substitute $x = e^t$ and $t = \ln|x|$ in this equation to obtain two independent solutions of $2x^2y'' + 4xy' + 3y = 0$:

$$x^{-1/2} \cos\left(\frac{\sqrt{5}}{2}\ln|x|\right), \quad e^{-t/2} \sin\left(\frac{\sqrt{5}}{2}\ln|x|\right).$$

Substitution Details. Because $x = e^t$, the factor $e^{-t/2}$ is written as $(e^t)^{-1/2} = x^{-1/2}$. Because $t = \ln|x|$, the trigonometric factors are back-substituted like this: $\cos\left(\frac{\sqrt{5}}{2}t\right) = \cos\left(\frac{\sqrt{5}}{2}\ln|x|\right)$.

General Solution. The final answer is the set of all linear combinations of the two preceding independent solutions.

Exercises 7.4

Cauchy-Euler Equation. Find solutions y_1, y_2 of the given homogeneous differential equation which are independent by the Wronskian test, page 452.

1. $x^2y'' + y = 0$

2. $x^2y'' + 4y = 0$

3. $x^2y'' + 2xy' + y = 0$

4. $x^2y'' + 8xy' + 4y = 0$

Variation of Parameters. Find a solution y_p using a variation of parameters formula.

5. $x^2y'' = e^x$

6. $x^3y'' = e^x$

7. $y'' + 9y = \sec 3x$

8. $y'' + 9y = \csc 3x$