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Math 3150 Problems
Haberman Chapter H2

Due Date: Problems are collected on Wednesday.

Chapter H2: 2.1–2.2 – Introduction, Linearity

Problem H2.2-2a. (Linear Operator)

Show that $L(u) = \frac{\partial}{\partial x}\left(K_0(x)\frac{\partial u}{\partial x}\right)$ is a linear operator.

Reference. Haberman H2.2.

Problem H2.2-4. (Superposition)

(a) Consider $L(u) = f$. If u_p is a particular solution, $L(u_p) = f$, and if u_l and u_2 are homogeneous solutions, $L(u_1) = L(u_2) = 0$, then show that $u = u_p + c_1u_1 + c_2u_2$ is another particular solution, that is, show that $L(u) = f$.

(b) If $L(u_1) = f_1$ and $L(u_2) = f_2$, then what is a particular solution u for $L(u) = f_1 + f_2$?

Problem XC-H2.2-5. (Generalized Superposition)

If L is a linear operator and $L(u_1) = L(u_2) = L(u_3) = 0$, then show that $L(c_1u_1 + c_2u_2 + c_3u_3) = 0$. Use this result to show that the principle of superposition may be extended to any finite number of homogeneous solutions.

Chapter H2: 2.3 – Separation of Variables Method

Problem H2.3-1. (Separated ODEs)

For the following partial differential equations, what ordinary differential equations are implied by the method of separation of variables?

(a) $\frac{\partial u}{\partial t} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$

(b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

(c) $\frac{\partial u}{\partial t} = k\frac{\partial^4 u}{\partial x^4}$

(d) $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$

Problem H2.3-2. (Eigenpairs)

Consider the differential equation $X'' + \lambda X = 0$.

Determine the eigenvalues λ and corresponding eigenfunctions if $X(x)$ satisfies the following boundary conditions. Analyze the three cases for lambda positive, zero and negative. You may assume that the eigenvalues are real.

(a) $X(0) = 0, X'(L) = 0$

(b) $X(a) = 0, X(b) = 0$ assuming λ positive

Problem H2.3-3. **(Rod Problem, Ice-pack Ends)**

Consider the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

subject to the boundary conditions

$$u(0,t) = 0 and u(L,t) = 0.$$

Solve the initial value problem if the temperature is initially $u(x,0) = 2\cos(3\pi x/L)$. Leave difficult integrals unevaluated.

Problem H2.3-5. **(Orthogonality, Sines)**

Evaluate (be careful if $n = m$) the integral over $0 < x < L$ of the product of $\sin(n\pi x/L)$ and $\sin(m\pi x/L)$ for $n > 0, m > 0$. Use the trigonometric identity $\sin(a)\sin(b) = (1/2)(\cos(a-b) - \cos(a+b))$, being careful if $a + b = 0$ or $a - b = 0$.

Problem H2.3-6. **(Orthogonality, Cosines)**

Evaluate the integral over $0 < x < L$ of the product of $\cos(n\pi x/L)$ and $\cos(m\pi x/L)$ for $n > 0, m > 0$. Use the trigonometric identity $\cos(a)\cos(b) = (1/2)(\cos(a+b) + \cos(a-b))$, being careful if $a + b = 0$ or $a - b$=0.

Problem H2.3-10. **(CSB Inequality)**

For two- and three-dimensional vectors, the fundamental property of dot products, $A \cdot B = |A||B|\cos(\theta)$, implies that

$$|A \cdot B| \leq |A||B|.$$

In this exercise we generalize this to n-dimensional vectors and functions, in which case $|A \cdot B| \leq |A||B|$ is known as Schwarz's inequality. [The names of Cauchy and Buniakovsky are also possible, the common name being the CSB inequality.]

(a) Show that $|A - \gamma B|^2 > 0$ implies $|A \cdot B| \leq |A||B|$, where $\gamma = A \cdot B / B \cdot B$.

(c) Generalize $|A \cdot B| \leq |A||B|$ to functions. [Hint: Let $A \cdot B$ mean the integral of $A(x)B(x)$ over x =0 to$x = L$.]