Name KEY

Math 3150 Midterm 1

19 Feb, S2014

150 Problem 1. **(Heat Conduction in a Rod, Ends at Different Temperatures)**
Throughout, k is the mean thermal diffusivity, usually written as Fourier's constant K_0 divided by specific heat c and mass density per unit volume ρ.

A (a) [40%] Consider the heat conduction problem in a laterally insulated rod of length 1 with one end at zero Celsius and the other end at 1 Celsius. The initial temperature along the rod is given by function $f(x) = x$.

$$\begin{cases} u_t &= ku_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) &= 0, \quad t > 0, \\ u(1,t) &= 1, \quad t > 0, \\ u(x,0) &= x, \quad 0 < x < 1. \end{cases}$$

The answer $u(x,t)$ to this problem is exactly the steady-state temperature. Find the answer $u(x,t)$ and display a complete answer check.

A (b) [60%] Consider the heat conduction problem in a laterally insulated rod of length 1 with one end at zero Celsius and the other end at one Celsius. The initial temperature along the rod is given by function $f(x) = 1 + x$.

$$\begin{cases} u_t &= ku_{xx}, \quad 0 < x < 1, \quad t > 0, \\ u(0,t) &= 0, \quad t > 0, \\ u(1,t) &= 1, \quad t > 0, \\ u(x,0) &= 1 + x, \quad 0 < x < 1. \end{cases}$$

Solve the rod problem for $u(x,t)$. It is necessary to derive the product solutions. Provide Fourier coefficient formulas. Evaluate all Fourier coefficients. Then display the final answer $u(x,t)$.

a) $\cancel{u_t}^{\,0} = k u_{xx} = 0$

$u = C_1 + C_2 x \qquad u(0,t) = 0 \qquad u(1,t) = 1$

$C_1 = 0 \qquad C_2 = 1 \Rightarrow \boxed{u(x,t) = x}$

plug into $u_t = k u_{xx}$

$u_t = 0$, $u_{xx} = 0$ $\therefore$ $0 = k(0) = 0$ ✓

b) $u(x,t) = X(x)\,T(t) \qquad X T' = k X'' T \Rightarrow \dfrac{T'}{kT} = \dfrac{X''}{X} = -\lambda$

$T' + k\lambda T = 0$

$X'' + \lambda X = 0$

T only $\lambda < 0$ makes sense

so $T = e^{-k\lambda t}$

X only $\lambda > 0$ makes sense

so $X = C_1 \sin(\sqrt{\lambda}\, x) + C_2 \cos(\sqrt{\lambda}\, x)$, $X(0) = 0$, $X(1) = 0$

$C_2 = 0 \qquad \sqrt{\lambda} = n\pi$

next page

steady state ...

problem 1(b) continued

Then $X = \sin(n\pi x)$ satisfies $X(0)=0$, $X(1)=0$ and
$T = e^{-n^2\pi^2 k t}$
form the product solutions for the ice-pack problem

$$\begin{cases} W_t = k W_{xx}, \\ W(0,t) = 0, \\ W(1,t) = 0, \\ W(x,0) = (1+x) - u_1(x) \\ \qquad = 1 \end{cases}$$

$u_1(x) = x$ is the steady-state from problem 1(a).

The solution u to problem 2(b) is then

$$u(x,t) = W(x,t) + u_1(x)$$

Because $W(x,t) = \sum_1^\infty a_n X_n T_n$, then

$$1 = W(x,0) = \sum_1^\infty a_n X_n(x) e^0$$

$$a_n = \frac{\vec{1} \cdot \vec{X}_n}{\vec{X}_n \cdot \vec{X}_n} = \frac{\int_0^1 1 \cdot \sin(n\pi x)\,dx}{\int_0^1 \sin^2(n\pi x)\,dx}$$

orthogonality used here

$$a_n = 2\int_0^1 \sin(n\pi x)\,dx = 2\left.\frac{\cos(n\pi x)}{n\pi}\right|_{x=0}^{x=1}$$

$$a_n = 2\left(\frac{(-1)^n - 1}{n\pi}\right)$$

Then

$$u(x,t) = W(x,t) + u_1(x)$$

$$u(x,t) = \sum_1^\infty \frac{2((-1)^n - 1)}{n\pi} \sin(n\pi x)\, e^{-n^2\pi^2 k t} + x$$

Problem 2. **(Total Thermal Energy in a Rod)**

An expression for the time-dependent total thermal energy contained in a rod $x = 0$ to $x = L$, with uniform cross-sectional area A is

$$\int_0^L c\rho u(x,t)\,A\,dx. \qquad 100$$

Symbol c is the specific heat, ρ is the mass density per unit volume and $u(x,t)$ is the rod temperature, satisfying the heat equation $u_t = k u_{xx}$. Assume c and ρ are constants.

Suppose $u(x,t)$ and $v(x,t)$ are two temperature distributions for the same rod which supply the same total thermal energy for all t.

A (a) [30%] Explain why $\int_0^L u(x,t)\,dx = \int_0^L v(x,t)\,dx$.

A (b) [60%] Differentiate on t across the equation of (a). Simplify the resulting equation using $u_t = k u_{xx}$ and $v_t = k v_{xx}$ to obtain

$$u_x(L,t) - u_x(0,t) = v_x(L,t) - v_x(0,t). \qquad \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

A ✓ (c) [20%] Explain the meaning of the equation in (b) in terms of heat flux and Fourier's Law.

Total Thermal Energy $= \cancel{c\rho A}\int_0^L u(x,t)\,dx = \cancel{c\rho A}\int_0^L v(t)\,dx$

a) This is because the rod is the same, so the values c, ρ, and A cancel out. The two integrals must be the same because they supply the same total thermal energy for all t, even though they are different functions.

b) $$\frac{\partial}{\partial t}\left(\int_0^L u(x,t)\,dx\right) = \frac{\partial}{\partial t}\left(\int_0^L v(x,t)\,dx\right)$$

$$\int_0^L \left(\frac{\partial}{\partial t}u(x,t)\right)dx = \int_0^L\left(\frac{\partial}{\partial t}v(x,t)\right)dx$$

$$\int_0^L u_t(x,t)\,dx = \int_0^L v_t(x,t)\,dx$$

$$\int_0^L k u_{xx}\,dx = \int_0^L k v_{xx}\,dx$$

$$\cancel{k}\int_0^L u_{xx}\,dx = \cancel{k}\int_0^L v_{xx}\,dx$$

$$u_x\Big]_0^L = v_x\Big]_0^L$$

$$\boxed{u_x(L,t) - u_x(0,t) = v_x(L,t) - v_x(0,t)}$$

c) Fourier's Law:

$\phi = -K_0 \dfrac{\partial u}{\partial t}$, ϕ = heat flux

This means that the difference between the heat flux at the right and left ends for $u(x,t)$ distribution must equal the heat flux at the right and left ends for the $v(x,t)$ temperature distribution

Problem 2. **(Total Thermal Energy in a Rod)**

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100

$$\int_0^L c\rho u(x,t)\,A\,dx.$$

Symbol c is the specific heat, ρ is the mass density per unit volume and $u(x,t)$ is the rod temperature, satisfying the heat equation $u_t = ku_{xx}$. Assume c and ρ are constants.

Suppose $u(x,t)$ and $v(x,t)$ are two temperature distributions for the same rod which supply the same total thermal energy for all t.

A (a) [30%] Explain why $\int_0^L u(x,t)\,dx = \int_0^L v(x,t)\,dx$.

A (b) [60%] Differentiate on t across the equation of (a). Simplify the resulting equation using $u_t = ku_{xx}$ and $v_t = kv_{xx}$ to obtain

$$u_x(L,t) - u_x(0,t) = v_x(L,t) - v_x(0,t).$$

A (c) [20%] Explain the meaning of the equation in (b) in terms of heat flux and Fourier's Law.

a. If same total thermal energy supplied:

$$\int_0^L c\rho u(x,t) A\,dx = \int_0^L c\rho v(x,t) A\,dx$$

$$c\rho A \int_0^L u(x,t)\,dx = c\rho A \int_0^L v(x,t)\,dx$$

properties of rod are the same, cancel out.

$$\int_0^L u(x,t)\,dx = \int_0^L v(x,t)\,dx$$

b. $$\frac{d}{dt}\int_0^L u(x,t)\,dx = \frac{d}{dt}\int_0^L v(x,t)\,dx$$

$$\int_0^L \frac{\partial u}{\partial t}(x,t)\,dx = \int_0^L \frac{\partial v}{\partial t}(x,t)\,dx$$

since $u_t = ku_{xx}$, $v_t = kv_{xx}$

$$k\int_0^L u_{xx}(x,t)\,dx = k\int_0^L v_{xx}(x,t)\,dx$$

using Fundamental Theorem of Calculus:

$$u_x(L,t) - u_x(0,t) = v_x(L,t) - v_x(0,t)$$

c. This indicates the heat flux, represented by u_x and v_x, is the same across the length of the rod regardless of temperature distribution.

problem 2(c). $\phi(x,t) = -K_0 u_x(x,t)$ (Fourier's Law)

$$K_0\left(u_x(L,t) - u_x(0,t)\right) = -\phi(L,t) + \phi(0,t)$$

= total heat flux from the ends $x=0$ & $x=L$

Heat only escapes from the ends (lateral insulation).

The equation means the total heat flux from the ends is the same for u and v, for each time $t>0$.

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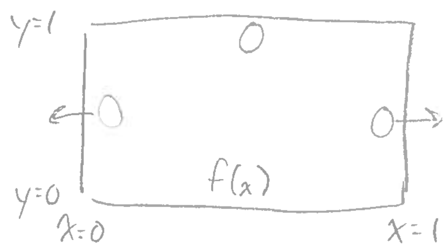
Problem 3. **(Steady-State Heat Conduction on a Rectangular Plate)**

Consider Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ on the rectangle $0 < x < 1$, $0 < y < 1$ subject to the boundary conditions $u_x(x, y) = 0$ for $x = 0$ and $x = 1$, $u(x, y) = 0$ for $y = 1$, $u(x, y) = f(x)$ for $y = 0$.

A- (a) [70%] Find the product solutions $u(x, y) = X(x)Y(y)$. Include a check that each product solution satisfies the required three zero boundary conditions.

A (b) [30%] Let $f(x)$ be the sum of the first three eigenfunctions, which is the sum of the first three X-answers. Find $u(x, y)$.

$$\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u_x(x,y) = 0 \text{ for } x = 0, 1 \\ u(x,y) = 0 \text{ for } y = 1 \\ u(x,y) = f(x) \text{ for } y = 0 \end{cases}$$



a) <u>product solutions:</u> $X(x)Y(y)$

$$\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{X''Y}{XY} = -\frac{XY''}{XY} \rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(1) = 0 \end{cases}$$

good

$$\begin{cases} Y'' - \lambda Y = 0 \\ Y(1) = 0 \end{cases}$$

$X'' + \lambda X = 0$

$\lambda = 0 \rightarrow X(x) = C_1 + C_2 x$

$X'(x) = C_2 \rightarrow \left.\begin{matrix} X'(0) = C_2 = 0 \\ X'(1) = C_2 = 0 \end{matrix}\right\} C_2 = 0$

$X(x) = C_1 \rightarrow \boxed{X = 1 \text{ for } \lambda = 0}$

$\lambda = 0 \rightarrow Y = \cancel{0}$ $y-1$

$\lambda > 0$

$\boxed{X(x) = \cos(\sqrt{\lambda}\, x) \text{ where } \sqrt{\lambda} = n\pi}$

by EPH 13.4 $n = 0, 1, 2 \ldots$

$\lambda > 0 \rightarrow Y(y) = C_1 e^{\sqrt{\lambda}\, y} + C_2 e^{-\sqrt{\lambda}\, y}$

$Y(1) = C_1 e^{\sqrt{\lambda}} + C_2 e^{-\sqrt{\lambda}} = 0$

$\boxed{Y(y) = \sinh(\sqrt{\lambda}(y - H)) \text{ for } \sqrt{\lambda} = n\pi}$

Correct!

Answer Checks on back

Problem 3. **(Steady-State Heat Conduction on a Rectangular Plate)** 87

Consider Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ on the rectangle $0 < x < 1$, $0 < y < 1$ subject to the boundary conditions $u_x(x,y) = 0$ for $x = 0$ and $x = 1$, $u(x,y) = 0$ for $y = 1$, $u(x,y) = f(x)$ for $y = 0$.

B+ (a) [70%] Find the product solutions $u(x,y) = X(x)Y(y)$. Include a check that each product solution satisfies the required three zero boundary conditions.

-8 B (b) [30%] Let $f(x)$ be the sum of the first three eigenfunctions, which is the sum of the first three X-answers. Find $u(x,y)$.

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$u_{xx} + u_{yy} = 0$ $\quad 0<x<1$, $0<y<1$ $\quad u_x(0) = u_x(1) = 0$, $u(x,1) = 0$, $u(x,0) = f(x)$

a) if $u = XY$, $\Rightarrow X''Y + YX'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \Rightarrow \begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases}$

for X

$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X(1) = 0 \end{cases}$

if $\lambda = 0$, $X = C_1 + C_2 x$, $X' = C_1$, $X'(0) = 0 = C_2 \Rightarrow C_2 = 0$ $\Rightarrow X_0 = 1$ ✓

if $\lambda > 0$, $X = C_1\cos(\sqrt{\lambda}x) + C_2\sin(\sqrt{\lambda}x)$

$X' = -C_1\sqrt{\lambda}\sin(\sqrt{\lambda}x) + C_2\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$X'(0) = 0 \Rightarrow C_2 = 0$ ✓

$X'(1) = 0 = \sin(\sqrt{\lambda}) \Rightarrow \sqrt{\lambda} = n\pi \Rightarrow X_n = \cos(n\pi x)$, $\sqrt{\lambda} = n\pi$, $n \geq 1$

for Y

$\begin{cases} Y'' - \lambda Y = 0 \\ Y(1) = 0 \\ \sqrt{\lambda} = n\pi \end{cases}$

if $\lambda = 0$, $Y = C_1 + C_2 y$

$Y(1) = 0 = C_1 + C_2(1) \Rightarrow -C_1 = C_2$ $\quad Y_0 = (1-y)$ good

if $\lambda > 0$ $\quad Y = C_1 e^{n\pi y} + C_2 e^{-n\pi y}$

$Y(1) = 0 = C_1 e^{n\pi} + C_2 e^{-n\pi} \Rightarrow Y = \cosh(n\pi y)$ No.

Answers

$Y = y - 1$ $(\lambda = 0)$

and

$Y = \sinh(n\pi(y-1))$

Good

$u_0(x,y) = X_0 Y_0 = a_0(1-y)$ checks $u_x = 0$ ✓ satisfied, $u(x,1) = 0$ ✓ satisfied

$u_n(x,y)$ $X_n Y_n = a_n\cos(n\pi x)\cosh(n\pi y)$

Not zero

Lcheck, @ $x = 0$ or $x = 1$, $\cos(n\pi x) = 0$ ✓ satisfied +

@ $y = 1$, $\cosh(n\pi) = 0$ ✓ satisfied ✗

Never zero

ans

8

solution check

[part (b) not printed here; see the next solution to problem 3.]

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Problem 3. **(Steady-State Heat Conduction on a Rectangular Plate)**

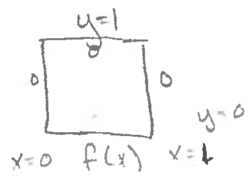
Consider Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ on the rectangle $0 < x < 1$, $0 < y < 1$ subject to the boundary conditions $u_x(x,y) = 0$ for $x = 0$ and $x = 1$, $u(x,y) = 0$ for $y = 1$, $u(x,y) = f(x)$ for $y = 0$.

Bt −8

(a) [70%] Find the product solutions $u(x,y) = X(x)Y(y)$. Include a check that each product solution satisfies the required three zero boundary conditions.

Bt −3

(b) [30%] Let $f(x)$ be the sum of the first three eigenfunctions, which is the sum of the first three X-answers. Find $u(x,y)$.



a) Using Laplace equation

$X''Y + Y''X$

$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$X'' + \lambda X = 0$

b.c. $X'(0)(Y)(Y) = 0$

$X'(1)(Y)(Y) = 0$

$X'(0) = 0$

$X'(1) = 0$

$\lambda > 0$

$X'' + \lambda X = 0$

$X = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}x$

b.c $X'(0) = 0$

$X'(1) = 0$

$X'(0) = \sqrt{\lambda}\, c_1 \sin\sqrt{\lambda}\,0 + \sqrt{\lambda}\, c_2 \cos\sqrt{\lambda}\,0 = 0$

$c_2 = 0$

$X'(1) = \sqrt{\lambda}\, c_1 \sin\sqrt{\lambda}\,1$

$\sqrt{\lambda} = n\pi$

$X = \cos\sqrt{\lambda}x$ ✓

$\lambda = 0$

$X'' = 0$

$X = c_1 x + c_2$

$X' = c_1 = 0$ ✓

$X = 1$

$Y'' - \lambda Y = 0$

B.C $\lambda > 0$

$X(x)\,Y(1) = 0$

$Pr = r^2 - \lambda \qquad r = \pm\sqrt{\lambda}$

$= e^{-1\sqrt{\lambda}} - e^{1\sqrt{\lambda}}$

B.C $Y(1) = 0$

$Y = \sinh\sqrt{\lambda}\,y$ ← Fails $u = 0$ at $y = 1$

$Y = c_1 e^{\sqrt{\lambda}y} + c_2 e^{-\sqrt{\lambda}y}$

$0 = c_1 e^{\sqrt{\lambda}} + c_2 e^{-\sqrt{\lambda}}$

$Y = c_1 e^{\sqrt{\lambda}y} - c_1 e^{2\sqrt{\lambda}} e^{-\sqrt{\lambda}y}$

$= 2c_1 e^{\sqrt{\lambda}} \sinh(\sqrt{\lambda}(y-1))$

Correction: $Y = \sinh(\sqrt{\lambda}(y-1))$

$\lambda = 0$

$Y'' = 0$

$Y = c_1 y + c_2$

$Y(1) = 0$ means $c_1 \cdot 1 + c_2 = 0$

$\cancel{c_2 = 0}$ $\quad c_2 = -c_1$, $Y = c_1(y-1)$

$Y = y$ ← Fails $u = 0$ at $y = 1$

Correction: $Y = y - 1$

Superposition

$\lambda > 0 \qquad \lambda = 0$

$X = \cos\sqrt{\lambda}x \qquad X = 1$

$\cancel{Y = \sinh\sqrt{\lambda}y} \qquad \cancel{Y = y}$

$Y = \sinh(\sqrt{\lambda}(y-1)) \qquad Y = y-1$

$u(x,y)\ \cancel{f(x)} = a_0 y + \sum_{n=1}^{\infty} a_n \cos\sqrt{\lambda}x \sinh\sqrt{\lambda}y$ ← Apply Corrections

Orthogonality (broken)

$\int_0^1 f(x)(1) = \int_0^1 a_0(0)^{(1)} + \sum_{n=1}^{\infty} a_n \cos\sqrt{\lambda}x \sinh\sqrt{\lambda}y$

$a_0 = \int_0^1 f(x)$

$\int_0^1 f(x)\cos m\pi x = a_m \sinh\sqrt{\lambda}(0) \int_0^1 \cos^2 m\pi x$ ↗ $\frac{1}{2}$

$a_m = \frac{2}{1}\int_0^1 f(x)\cos m\pi x\,dx$

...blem 3b

$$u(x,y) = a_0(1-y) + \sum_1^{\infty} a_n \cos(n\pi x) \cancel{\cosh(n\pi y)}$$ $\sinh(n\pi(y-1))$

$$f(x) = 1 + \cos(\pi x) + \cos(2\pi x)$$ good

$$u(x,0) = f(x) = a_0 + \sum_1^{\infty} a_n \cos(n\pi x)$$

$$\boxed{\begin{gathered} a_0 = \frac{\int_0^1 f(x)\,dx}{\int_0^1 (1)(1)\,dx} \\ \sinh(-n\pi)\, a_n = \frac{\int_0^1 f(x)\cos(n\pi x)\,dx}{\int_0^1 \cos^2(n\pi x)\,dx} \quad n \geq 1 \\ u(x,y) = a_0(1-y) + \sum_1^{\infty} a_n \cos(n\pi x) \cancel{\cosh(n\pi y)} \end{gathered}}$$

ans

please, insert y-terms into $f(x)$.
No Fourier coefficient formula is needed

Solution

$$u(x,y) = 1 \cdot \frac{Y_0(y)}{Y_0(0)} + \cos(\pi x)\frac{Y_1(y)}{Y_1(0)} + \cos(2\pi x)\frac{Y_2(y)}{Y_2(0)}$$

$$Y_0 = y-1, \quad Y_1 = \sinh(\pi(y-1)), \quad Y_2 = \sinh(2\pi(y-1))$$

Then

$$u(x,0) = 1 + \cos(\pi x) + \cos(2\pi x) = f(x)$$

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Problem 4. **(Steady-State Heat Conduction on a Disk)**

Consider the steady-state heat conduction problem in polar coordinates

$$\begin{cases} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, & 0 < r < 2, \quad 0 < \theta < 2\pi, \\ u(2,\theta) = f(\theta), & 0 < \theta < 2\pi. \end{cases}$$

A (a) [60%] Find the product solutions $u = R(r)\Theta(\theta)$, then identify the orthogonal set and the interval. Stop at this stage: omit superposition, omit the series solution and do not develop formulas for the Fourier coefficients.

A (b) [40%] Calculate $u(0,\theta)$ when $f(\theta) = 0$ on $0 \le \theta < \pi$, $f(\theta) = 50$ on $\pi \le \theta < 2\pi$. Hint: The Poisson integral theorem and the Mean Value Theorem.

a) from experience,

$$\boxed{\begin{aligned} &u_0(r,\theta) = a_0 \\ &u_n(r,\theta) = [a_n \cos(n\pi) + b_n \sin(n\pi)]r^n \\ &\text{Orthogonal set} = \{1, \cos(n\pi), \sin(n\pi)\} \\ &\text{interval of } 0 < \theta < 2\pi \end{aligned}}$$

ans

b) $f(\theta) = 0$ on $0 \le \theta \pi$

$f(\theta) = 50$ $\pi \le \theta < 2\pi$

Mean value Theorem states: @ $r = 0$, $u = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$, where $f(\theta)$ is the temperature distribution along the edge of the disk

$\therefore$ $u(0,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_\pi^{2\pi} (50)\,d\theta = \frac{1}{2\pi}(50\pi) = 25$

$\boxed{u(0,\theta) = 25}$ ans