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Partial Differential Equations 3150

Midterm Exam 2

Exam Date: Tuesday, 1 December 2009

Instructions: This exam is timed for 50 minutes. You will be given double time to complete the exam. No calculators, notes, tables or books. Problems use only chapters 1 to 4 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

100
100
100
100

1. (CH3. Finite String: Fourier Series Solution)

100

The series formula for

$$\begin{cases} u_{tt}(x,t) = c^2 u_{xx}(x,t), & 0 < x < \pi, \quad t > 0, \\ u(0,t) = 0, & t > 0, \\ u(\pi,t) = 0, & t > 0, \\ u(x,0) = f(x), & 0 < x < \pi, \\ u_t(x,0) = 0, & 0 < x < \pi \end{cases}$$

has the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \cos(nct)\sin(nx)$$

where $\{a_n\}$ are Fourier coefficients of the odd periodic function $f(x)$.

(a) [50%] Display an explicit formula for the Fourier coefficients which contains the symbol $f(x)$.

(b) [25%] Compute all the Fourier coefficients when $f(x) = 2\sin(x) - 5\sin(7x)$.

(c) [25%] Display the solution $u(x,t)$ when $c = 1/\pi$ and $f(x) = 2\sin(x) - 5\sin(7x)$.

Use this page to start your solution. Attach extra pages as needed, then staple.

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2. (CH3. Heat Conduction in a Bar)

Consider the heat conduction problem in a laterally insulated bar of length 1 with one end at 100 Celsius and the other end at zero Celsius.

100

$$\begin{cases} u_t = c^2 u_{xx}, & 0 < x < 1, \quad t > 0, \\ u(0,t) = 100, & t > 0, \\ u(1,t) = 0, & t > 0, \\ u(x,0) = x(1-x), & 0 < x < 1. \end{cases}$$

(a) [25%] Find the steady-state temperature $u_1(x)$.

(b) [75%] Solve the bar problem for $u(x,t)$, but to save time don't evaluate any integrals.

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3. (CH4. Rectangular Membrane)
Consider the special membrane problem

$$\begin{cases} u_{tt}(x,y,t) &= (1/\pi^2)\,(u_{xx}(x,y,t)+u_{yy}(x,y,t)), & 0<x<1,\ \ 0<y<1,\ \ t>0, \\ u(x,y,t) &= 0 & \text{on the boundary}, \\ u(x,y,0) &= 1, & 0<x<1,\ \ 0<y<1, \\ u_t(x,y,0) &= 0, & 0<x<1,\ \ 0<y<1. \end{cases}$$

Solve the problem for $u(x,t)$ in series form, but to save time leave nonzero integrals unevaluated. Display the shortest expression you can determine, by removing obviously zero terms. Details about product solutions should be sketched, in order to save writing time.

More expected for B_{mn}

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4. **(CH4. Steady-State Heat Conduction on a Disk)**
Consider the problem

100

$$\begin{cases} u_{rr}(r,\theta) + \frac{1}{r}u_r(r,\theta) + \frac{1}{r^2}u_{\theta\theta}(r,\theta) = 0, & 0 < r < a, \quad 0 < \theta < 2\pi. \\ u(a,\theta) = f(\theta), & 0 < \theta < 2\pi. \end{cases}$$

It is known that

$$u(r,\theta) = \sum_{n=0}^{\infty} \left(\frac{r}{a}\right)^n (a_n \cos(n\theta) + b_n \sin(n\theta)).$$

✓ (a) [50%] A product solution has the form $R(r)\Theta(\theta)$. Assume the result that $\Theta'' + n^2\Theta = 0$ and $n = 1, 2, 3, \ldots$, which implies $\Theta(\theta)$ equals $a_n \cos(n\theta) + b_n \sin(n\theta)$. Justify the formula $R(r) = (r/a)^n$.

✓ (b) [50%] Find explicitly the Fourier coefficients when $a = 1$ and $f(\theta) = \frac{1}{2}(\pi - \theta)$ on $0 < \theta < 2\pi$, $f(0) = f(2\pi) = 0$, $f(\theta + 2\pi) = f(\theta)$. To simplify the integrations, use the differentiation formulas $\frac{d}{d\theta}\sin\theta = \cos\theta$, $\frac{d}{d\theta}\cos\theta = -\sin\theta$, $\frac{d}{d\theta}(\sin\theta - \theta\cos\theta) = \theta\sin\theta$.

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a)

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

b)

$f(x) = 2\sin(x) - 5\sin(7x)$

so

$a_1 = 2$

$a_7 = -5$

all other $a_n = 0$

c)

$$u(x,t) = 2\cos\left(\frac{nt}{\pi}\right)\sin x - 5\cos\left(\frac{nt}{\pi}\right)\sin(7x)$$

a) steady-state

$u_1 = \frac{T_2 - T_1}{L}x + T_1$

$= 0 - 100x + 100$

$= 100(1-x)$

b) $X'' + \mu^2 X = 0$ Let $u_2(0,t) = u_2(1,t) = 0$

$T' + c^2\mu^2 T = 0$ and $u_2(x,0) = f(x) - u_1$

$X = c_1 \cos \mu x + c_2 \sin \mu x$

$X(0) = X(1) = 0$

$c_1 = 0, \; \mu = n\pi, \; n = 1, 2, \ldots$

$T = ke^{-\lambda_n t}$ where $\lambda_n = (cn\pi)^2$

$u_2 = X(x)T(t)$

$= \sum_{n=1}^{\infty} b_n \sin n\pi x \cdot e^{-\lambda_n t}$

where $b_n = 2\int_0^1 (x(1-x) - 100(1-x)) \sin n\pi x \, dx$

the final solution is

$u = u_1 + u_2$

$= 100(1-x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x) \cdot e^{-\lambda_n t}$

)

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$$T'' + \frac{1}{\pi^2}k^2 T = 0$$

$$X'' + \mu^2 X = 0$$

$$Y'' + \nu^2 Y = 0 \qquad k^2 = \mu^2 + \nu^2$$

so

$$X = c_1 \cos \mu x + c_2 \sin \mu x$$

$$Y = c_1 \cos \nu y + c_2 \sin \nu y$$

B.C.

$$X(0) = X(1) = 0$$

$$Y(0) = Y(1) = 0$$

so

$$c_1 = 0 \qquad \mu = m\pi$$

$$\nu = n\pi$$

$$T = c_1 \cos ckt + c_2 \sin ckt$$

Let

$$\lambda_{mn} = \frac{1}{\pi} \cdot k = \frac{1}{\pi}\sqrt{\mu^2 + \nu^2} = \frac{\pi}{\pi}\sqrt{m^2 + n^2} = \sqrt{m^2 + n^2}$$

so

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} (B_{mn} \cos \lambda_{mn} t + B^*_{mn} \sin \lambda_{mn} t) \sin m\pi x \sin n\pi y$$

$$B_{mn} = 4\int_0^b \int_0^a \sin m\pi x \sin n\pi y \, dx\, dy = \begin{cases} 0 & m,n \text{ even} \\ \neq 0 & m,n \text{ odd} \end{cases}$$

$$B^*_{mn} = 0$$

so

$$u(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn} \cos \lambda_{mn} t \sin m\pi x \sin n\pi y$$

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4) a) $u(r,\theta) = R(r)\Theta(\theta)$

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$$

$$= R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$$

so

$$r^2\frac{R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \mu$$

we know $\mu = n^2$ from the Θ equation so

$$r^2R'' + rR' - n^2R = 0$$

Let

$$\begin{cases} r = e^t \\ z(t) = R(r) \end{cases}$$

$$(z'' - z') + z' - n^2 z = 0$$

$$z'' - n^2 z = 0$$

so

$$z = c_1 e^{nt} + c_2 e^{-nt}$$

or

$$= c_1(e^t)^n + c_2(e^t)^{-n}$$

so

$$R(r) = c_1\left(\frac{r}{a}\right)^n + c_2\left(\frac{r}{a}\right)^{-n}$$

since $\left(\frac{r}{a}\right)^{-n} \to \infty$ as $r \to 0$

we choose $c_2 = 0$

so

$$R(r) = \left(\frac{r}{a}\right)^n$$

✓

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1b)

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta \, d\theta$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta \, d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \pi\cos n\theta - \theta\cos n\theta \, d\theta$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n}\sin n\theta - \left(\frac{\cos n\theta}{n^2} + \frac{\theta \sin n\theta}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n}\sin n2\pi - \frac{\cos n2\pi}{n^2} - \frac{2\pi \sin n2\pi}{n} - \frac{\pi}{n}\sin 0 + \frac{\cos 0}{n^2} + 0\right]$$

$$= \frac{1}{2\pi}\left(0 - \frac{1}{n^2} - 0 - 0 + \frac{1}{n^2}\right)$$

$$= 0$$

$$b_n = \frac{1}{2\pi}\int_0^{2\pi} \pi\sin n\theta - \theta\sin n\theta \, d\theta$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{n}\cos n\theta - \left(\frac{\sin n\theta}{n^2} - \frac{\theta\cos n\theta}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\pi}{n}\cos n2\pi - \frac{\sin n2\pi}{n^2} + \frac{2\pi\cos n2\pi}{n} + \frac{\pi}{n}\cos 0 + \frac{\sin 0}{n^2} - 0\right]$$

$$= \frac{1}{2\pi}\left(-\frac{\pi}{n} - 0 + \frac{2\pi}{n} + \frac{\pi}{n} + 0\right)$$

$$= \frac{1}{2\pi}\left(\frac{2\pi}{n}\right) = \frac{1}{n}$$

$$\frac{d}{d\theta}(\sin\theta - \theta\cos\theta)$$

$$= \cos\theta - \left(\frac{d\theta}{d\theta}\cos\theta + \theta\frac{d\cos\theta}{d\theta}\right)$$

$$= \cos\theta - (\cos\theta - \theta\sin\theta)$$

= cos θ + [illegible] + θ sin θ

$$\frac{d}{d\theta}\left(\frac{\cos n\theta}{n^2} + \theta\frac{\sin n\theta}{n}\right)$$

$$= -\frac{\sin n\theta}{n} + \left(\frac{d\theta}{d\theta}\frac{\sin n\theta}{n} + \theta\frac{d\sin n\theta}{n}\right)$$

$$= -\frac{\sin n\theta}{n} + \frac{\sin n\theta}{n} + \theta\cos n\theta$$

$$= \theta\cos n\theta$$