Partial Differential Equations 3150

Sample Final Exam
Exam Date: Friday, April 25, 2014

Instructions: This exam is timed for 120 minutes. You may be given extra time to complete the exam (10-15 extra minutes). No calculators, notes, tables or books. Problems use topics from the required textbook which were covered in lectures. Details count 3/4, answers count 1/4.

1. (Chapters H1-H2. Heat Conduction in a Bar)

Considered is the heat conduction problem

$$\begin{cases} u_t &= \frac{1}{4}u_{xx}, \quad 0 < x < 10, \quad t > 0, \\ u(0,t) &= 50, \qquad\qquad\qquad\quad t > 0, \\ u(10,t) &= 80, \qquad\qquad\qquad\quad t > 0, \\ u(x,0) &= f(x), \quad 0 < x < 10. \end{cases} \tag{1}$$

It represents a laterally insulated uniform bar of length 10 with one end at 50 Celsius and the other end at 80 Celsius, and initial temperature $f(x)$.

1(a) [30%] Show the details for finding the the steady-state temperature $u_1(x) = 50 + 3x$.

1(b) [40%] Show the details for the solution $u_2(x,t) = \sin(\pi x/10)\, e^{-\pi^2 t/400}$ of the ice-pack ends bar problem

$$\begin{cases} u_t &= \frac{1}{4}u_{xx}, \qquad\qquad 0 < x < 10, \quad t > 0, \\ u(0,t) &= 0, \qquad\qquad\qquad\qquad\qquad\quad t > 0, \\ u(10,t) &= 0, \qquad\qquad\qquad\qquad\qquad\quad t > 0, \\ u(x,0) &= \sin(\pi x/10), \quad 0 < x < 10. \end{cases}$$

1(c) [30%] Superposition implies $u(x,t) = u_1(x,t) + u_2(x,t) = 50 + 3x + \sin(\pi x/10)\, e^{-\pi^2 t/400}$ is a solution of (1) with $f(x) = -50 - 3x + \sin(\pi x/10)$. Show the details of an answer check for this solution $u(x,t)$.

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2. (Chapter H3. Fourier Series)

2(a) [20%] Find and display the nonzero terms in the Fourier series expansion of $f(x)$, formed as the even 2π-periodic extension of the function $f_0(x) = \sin^2(x) + 4\cos(2x)$ on $0 < x < \pi$.

2(b) [40%] Let $g(x)$ be the Fourier sine series for the the period 2 odd extension of the function $g_0(x) = 1$ on $0 \le x \le 1$. Complete the following.

- **(1)** Graph of g_0 with its odd extension on $|x| < 1$.
- **(2)** Graph of $g(x)$ over 4 periods.
- **(3)** Fourier sine series coefficient formulas.
- **(4)** Numerical values for the coefficients.
- **(5)** Gibb's overshoot graphic on $|x| < 2$.

2(c) [20%] Define $h_0(x) = \begin{cases} \sin(2x) & 0 \le x < \pi, \\ x - \pi & \pi \le x \le 2\pi, \end{cases}$ and let $h(x)$ be the 4π odd periodic extension of $h_0(x)$ to the whole real line. Compute the sum $h(-5.25\pi) + h(1.5\pi)$.

2(d) [20%] Compute the smallest period, for those functions which are periodic.

- **(1)** $f_1(x) = 2\sin(x) + 5\cos(x)$
- **(2)** $f_2(x) = \sin(x) + \cos(5x)$
- **(3)** $f_3(x) = \sin(x) + \cos(\pi x)$
- **(4)** $f_4(x) = \sin(x) + \sin(x)\cos(x)$

3. (CH H4. Finite String: Fourier Series Solution)

3(a) [50%] Complete the following for the finite string problem

$$\begin{cases} u_{tt}(x,t) &= \frac{1}{4}u_{xx}(x,t), & 0 < x < 2, & t > 0, \\ u(0,t) &= 0, & & t > 0, \\ u(2,t) &= 0, & & t > 0, \\ u(x,0) &= f(x), & 0 < x < 2, & \\ u_t(x,0) &= g(x), & 0 < x < 2. & \end{cases}$$

(1) Separation of variables details.

(2) Product solutions and boundary conditions.

(3) The normal modes.

(4) Superposition.

(5) Series solution $u(x,t)$.

3(b) [25%] Display explicit formulas for the generalized Fourier coefficients which contains the symbols $f(x)$, $g(x)$.

3(c) [25%] Evaluate the coefficients when $f(x) = 100$ and $g(x) = 0$.

Use this page to start your solution. Attach any extra pages, then re-staple.

4. (CH H4. Rectangular Membrane)
Complete the following for the general membrane problem

$$\begin{cases} u_{tt}(x,y,t) &= c^2\left(u_{xx}(x,y,t)+u_{yy}(x,y,t)\right), & 0<x<a, \quad 0<y<b, \quad t>0, \\ u(x,y,t) &= 0 & \text{on the boundary}, \\ u(x,y,0) &= f(x,y), & 0<x<a, \quad 0<y<b, \\ u_t(x,y,0) &= g(x,y), & 0<x<a, \quad 0<y<b. \end{cases}$$

under the assumptions $a=b=c=1$, $f(x,y)=1$, $g(x,y)=0$.

(1) Separation of variables.
(2) Product solution boundary value problems.
(3) Product solutions (the normal modes).
(4) Superposition for $u(x,y,t)$.
(5) Generalized Fourier coefficient formulas.
(6) Explicit numerical values for the coefficients.

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5. (CH H5. Heat Equation and Gauss' Heat Kernel)

5(a) [10%] Define a Fourier transform pair so as to include as many definitions as possible, in particular, Haberman's definition.

5(b) [20%] Assume $f(x) = 100$ on $-1 \le x \le 1$ and $f(x) = 0$ otherwise. Compute the Fourier transform $F(w)$ of $f(x)$.

Graded Details: (1) Transform formula, (2) Integration details, (3) Answer.

5(c) [50%] The heat kernel $g(x)$ and the error function $\mathbf{erf}(x)$ are defined by the equations

$$g(x) = \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}, \quad \mathbf{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz.$$

Solve the infinite rod heat conduction problem

$$\begin{cases} u_t(x,t) &= \dfrac{1}{16} u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), & -\infty < x < \infty, \\ f(x) &= \begin{cases} 50 & 0 < x < 1, \\ 100 & -1 < x < 0, \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

Graded Details: (1) Fourier Transform method, (2) Convolution, (3) Heat kernel use, (4) Error function methods, (5) Final answer, expressed in terms of the error function.

5(d) [10%] Compute the limits at x equal to infinity and minus infinity of $u(x,t)$.

5(e) [10%] Compute the limit $u(x, 0+)$ for each x in $-\infty < x < \infty$.

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