# Partial Differential Equations 3150

**Final Exam**

**Exam Date: Thursday, May 2, 2013**

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**Instructions:** This exam is timed for 120 minutes. You will be given extra time to complete the exam. No calculators, notes, tables or books. Problems use only Chapters 1, 2, 3, 4, 7 of the textbook. No answer check is expected. Details count 3/4, answers count 1/4.

100 **1. (CH3. Heat Conduction in a Bar)**

Consider the heat conduction problem in a laterally insulated bar of length 10 with one end at 50 Celsius and the other end at 80 Celsius. The initial temperature along the bar is $f(x) = 3x$.

$f(x) = 3x$

$$\begin{cases} u_t = \frac{1}{4} u_{xx}, & 0 < x < 10, \quad t > 0, \\ u(0,t) = 50, & t > 0, \\ u(10,t) = 80, & t > 0, \\ u(x,0) = f(x), & 0 < x < 10. \end{cases}$$

$L = 10$

A (a) [20%] Find the steady-state temperature $u_1(x)$.

A (b) [40%] Solve the bar problem with zero Celsius temperatures at both ends, but $f(x)$ replaced by $f(x) - u_1(x)$. Call the answer $u_2(x,t)$.

A (c) [20%] Compute the numerical values of the Fourier coefficients of $u_2(x,t)$. Then display the answer $u(x,t) = u_1(x) + u_2(x,t)$ with the numerical values inserted into the series.

A (d) [20%] Display an answer check for the solution $u(x,t) = u_1(x) + u_2(x,t)$.

a) STEADY STATE $u_1(x) = c_1 x + c_2$

$u(0) = 50 = c_1(0) + c_2 \Rightarrow c_2 = 50$

$u(10) = 80 = c_1(10) + 50 \Rightarrow c_1 = 3$

$u_1(x) = 3x + 50$ ✓

b) $u = XT$

$u_t = XT'$

$u_{xx} = X''T$

$c^2 = \frac{1}{4} \Rightarrow \frac{\cancel{X}T'}{c^2 \cancel{X} T} = \frac{\cancel{c^2} X'' \cancel{T}}{\cancel{c^2} X \cancel{T}} \Rightarrow \frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda^2$

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$\Rightarrow X'' + \lambda^2 X = 0$

$X(x) = a_1 \cos(\lambda x) + a_2 \sin(\lambda x)$

$X(0) = 0 = a_1 \cos(0) + a_2 \sin(0)$

$\therefore a_1 = 0$

$X(t) = a_2 \sin \lambda x = a_2 \sin\left(\frac{n\pi}{10} x\right)$

$X(10) = 0 = a_2 \sin(10\lambda)$

$10\lambda = n\pi$

$\therefore \lambda = \frac{n\pi}{10}$

$$T' + \lambda^2 c^2 T = 0$$

$$T(t) = a_3 e^{-\lambda^2 c^2 t} = a_3 e^{-\left(\frac{n\pi}{10}\right)^2 \frac{1}{4} t} = a_3 e^{-\left(\frac{n\pi}{20}\right)^2 t}$$

$$u_2(x,t) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{10}x\right) e^{-\left(\frac{n\pi}{20}\right)^2 t} = f(x) - u_1(x)$$

$\hookrightarrow u(x,0) = f(x) = 3x = u_1(x) + u_2(x,0) = 3x + 50 + \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{10}x\right)$

MULTIPLY BOTH SIDES BY $\sin \frac{n\pi}{10}x$ AND INTEGRATE OVER $0 \to L$

$$\left[\cancel{3x} - \cancel{3x} + 50 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{10}x\right)\right] \sin \frac{n\pi}{10} x$$

$$50 \int_0^{10} \sin\left(\frac{n\pi}{10}x\right) dx = \frac{L}{2} a_n$$

$L = 10$

$$a_n = 10 \int_0^{10} \sin\left(\frac{n\pi}{10}x\right) dx = \frac{-100}{n\pi} \cos\left(\frac{n\pi}{10}x\right)\Big|_0^{10}$$

$$a_n = \frac{-100}{n\pi}\left[\cos(n\pi) - 1\right] \quad \text{WHERE } n = 1, 2, 3, \ldots$$

$$a_n = \begin{cases} \frac{200}{n\pi} & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN} \end{cases}$$

d) CHECK:

1) $u(0,t) = u_1(0) + u_2(0,t) = 0 + 50 + 0 = 50$ ✓

2) $u(10,t) = u_1(10) + u_2(10,t) = 3(10) + 50 + 0 = 80$ ✓

3) $u(x,0) = u_1(x) + \underset{f(x) - u_1(x) =}{u_2(x,0)} = \cancel{3x} + \cancel{50} + [3x - \cancel{3x} + \cancel{50}] = 3x = f(x)$ ✓

4) $u_t = \frac{1}{4} u_{xx} \Rightarrow$ $u(x,t) = u_1(x) + u_2(x,t)$

$$\partial_t u = \partial_t u_1 + \partial_t u_2$$

$$u_t = 0 + \partial_t u_2$$

$$= 0 + \frac{1}{4} \partial_x \partial_x u_2$$

$$= \frac{1}{4} u_{xx}$$

$\Rightarrow u_t = \frac{1}{4} u_{xx}$

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A (a) [20%] Find the steady-state temperature $u_1(x)$.

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A (c) [20%] Compute the numerical values of the Fourier coefficients of $u_2(x,t)$. Then display the answer $u(x,t) = u_1(x) + u_2(x,t)$ with the numerical values inserted into the series.

A (d) [20%] Display an answer check for the solution $u(x,t) = u_1(x) + u_2(x,t)$.

a) $U_t = 0 \Rightarrow U = C_1 x + C_2$

$U(0) = 50 \Rightarrow C_2 = 50 \quad U(10) = 80 \Rightarrow 80 = 10C_1 + 50$

$\Rightarrow C_1 = 3 \quad \boxed{U_1(x) = 3x + 50}$ ✓

b) $U_2(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{10}x\right) e^{-\left(\frac{n\pi}{20}\right)^2 t}$

$b_n = \frac{1}{5}\int_0^{10} (\cancel{3x} - \cancel{3x} - 50)\sin\left(\frac{n\pi}{10}x\right)dx$ ✓ ok

c) $b_n = -10\int_0^{10} \sin\left(\frac{n\pi}{10}x\right)dx = -10\left(-\frac{10}{n\pi}\right)\left(\cos\left(\frac{n\pi}{10}x\right)\right)\Big|_0^{10}$

$= \frac{100}{n\pi}(\cos(n\pi) - 1) = \frac{100}{n\pi}((-1)^n - 1) = \begin{cases} 0 \text{ if } n \text{ is even} \\ \frac{-200}{n\pi} \text{ if } n \text{ is odd} \end{cases}$

Use this page to start your solution. Attach extra pages as needed, then staple.

Then $U(x,t) = 3x + 50 - \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi}{10}x\right) \cdot e^{-\left(\frac{(2n-1)\pi}{20}\right)^2 t}$

D) $U_t = -\left(\frac{(2n-1)\pi}{20}\right)^2 U_2, \quad \frac{1}{4}U_{xx} = -\frac{1}{4}\left(\frac{(2n-1)\pi}{10}\right)^2 U_2 = -\left(\frac{(2n-1)\pi}{20}\right)^2 U_2$

note $U_t = \frac{1}{4}U_{xx}$, Thus $U(x,t)$ is a valid solution!

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**2. (Fourier Series)**

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$\cos(2\theta) = 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$

(a) [30%] Find and display the nonzero terms in the Fourier series expansion of $f(x)$, formed as the even $2\pi$-periodic extension of the function $f_0(x) = \sin^2(x) + 4\cos(2x)$ on $0 < x < \pi$.

(b) [50%] Compute the Fourier sine series coefficients $b_n$ for the function $g(x)$, defined as the period 2 odd extension of the function $g_0(x) = 1$ on $0 \le x \le 1$. Draw a representative graph for the partial Fourier sum for five terms of the infinite series.

$T = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$

(c) [20%] Define $h_0(x) = \begin{cases} \sin(2x) & 0 \le x < \pi, \\ x - \pi & \pi \le x \le 2\pi, \end{cases}$ and let $h(x)$ be the $4\pi$ odd periodic extension of $h_0(x)$ to the whole real line. Compute the sum $f(-5.25\pi) + f(1.5\pi)$.

a) $f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$

$\cancel{4\cos(2x) = 4(1-2\sin^2(x))}$

$\cancel{f_0(x) = \sin^2 x + 4 - 8\sin^2(x)}$

$\cancel{f_0(x) = 4 - 7\sin^2(x)}$

$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2} + \frac{7}{2}\cos(2x)\,dx$

$f_0(x) = \frac{1-\cos(2x)}{2} + 4\cos(2x)$

$a_0 = \frac{1}{2\pi}\left(\frac{1}{2}x + \frac{7}{4}\sin(2x)\right)\Big|_{-\pi}^{\pi}$

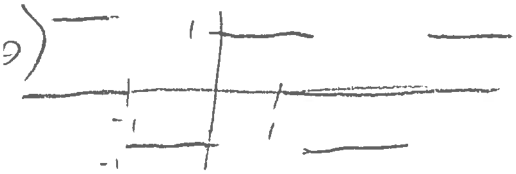
$f_0(x) = \frac{1}{2} + \frac{7}{2}\cos(2x)$

$a_0 = \frac{1}{2\pi}\left(\frac{\pi}{2} + \frac{\pi}{2}\right)$ $a_0 = \frac{1}{2}$

$\int_{-\pi}^{\pi}\left(\frac{1}{2} + \frac{7}{2}\cos 2x\right)\cos(2x) = a_n\int_{-\pi}^{\pi}\cos^2(2x)\,dx$ $b_{n_s} = 0$

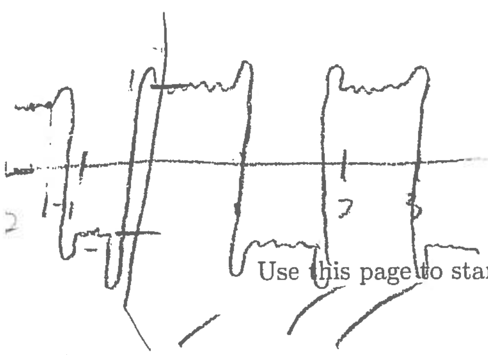
$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\cos(2x)}{2} + \frac{7}{2}\cos^2(2x)\,dx$ $a_n = \frac{7}{2\pi}\int_{-\pi}^{\pi}\cos^2(2x) = \frac{7\pi}{2\pi} = \frac{7}{2}$

$a_2 = \frac{7}{2}$

b) 

$f(x) = a_0 + \sum a_n\cos\left(\frac{n\pi x}{1}\right) + b_n\sin(n\pi x)$

$\int_{-1}^{1} g(x)\sin(n\pi x)\,dx = b_n\int_{-1}^{1}\cancel{\sin^2(n\pi x)}$



$b_n = \int_0^1 \sin(n\pi x)\,dx - \int_{-1}^{0}\sin(n\pi x)\,dx = \frac{-1}{n\pi}(\cos(n\pi x))\Big|_0^1 + \frac{1}{n\pi}\cos(n\pi x)\Big|_{-1}^{0}$

$b_n = -\frac{1}{n\pi}((-1)^n - 1) + \frac{1}{n\pi}(1 - (-1)^n) = \frac{-2}{n\pi}((-1)^n - 1)$

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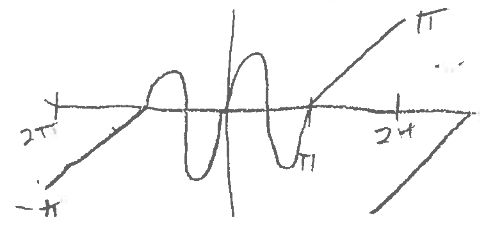
$= \frac{4}{n\pi}$ for odd n's

Gibbs phenomenon at discontinuities

c) 


$f(-5.25\pi) = f(-1.25\pi) = -1.25\pi + \pi = -.25\pi$

$f(1.5\pi) = 1.5\pi - \pi = .5\pi$

$f(-5.25\pi) + f(1.5\pi) = -.25\pi + .5\pi = .25\pi$

**3. (CH3. Finite String: Fourier Series Solution)**

(a) [50%] Display the series formula, **complete with derivation details**, for the solution $u(x,t)$ of the finite string problem

$$\begin{cases} u_{tt}(x,t) &= \frac{1}{4}u_{xx}(x,t), & 0<x<2, & t>0,\\ u(0,t) &= 0, & & t>0,\\ u(2,t) &= 0, & & t>0,\\ u(x,0) &= f(x), & 0<x<2, & \\ u_t(x,0) &= g(x), & 0<x<2. & \end{cases}$$

$c^2 = \frac{1}{4}$

Symbols $\cancel{f}$ and $\cancel{g}$ should not appear explicitly in the series for $u(x,t)$. Expected in the formula for $u(x,t)$ are product solutions times constants.

(b) [25%] Display explicit formulas for the Fourier coefficients which contains the symbols $f(x)$, $g(x)$.

(c) [25%] Evaluate the Fourier coefficients when $f(x) = 100$ and $g(x) = 0$.

A

a) $u = XT \qquad u_{tt} = XT'' \qquad u_{xx} = X''T$

$\lambda c = \frac{n\pi}{2}\left(\frac{1}{2}\right) = \frac{n\pi}{4}$

$\therefore \frac{\cancel{X}T''}{c^2\cancel{X}T} = \frac{\cancel{c^2}X''\cancel{T}}{\cancel{c^2}X\cancel{T}} \Rightarrow \frac{T''}{c^2T} = \frac{X''}{X} = -\lambda^2$

$X'' + \lambda^2 X = 0$

$X(x) = a_1\cos(\lambda x) + a_2\sin(\lambda x)$

$X(0) = 0 = a_1 \cancel{\cos 0}^{1} + \cancel{a_2 \sin(0)}^{0}$

$\therefore a_1 = 0$

$X(2) = 0 = a_2 \sin(2\lambda)$

$2\lambda = n\pi \Rightarrow \lambda = \frac{n\pi}{2}$

$X(x) = a_2 \sin\left(\frac{n\pi}{2}x\right)$

$T'' + \lambda^2 c^2 = 0$

$T(t) = a_3\cos(\lambda c t) + a_4 \sin(\lambda c t)$

$T(t) = a_3 \cos\left(\frac{n\pi}{4}t\right) + a_4\sin\left(\frac{n\pi}{4}t\right)$

✓ $u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}x\right)\left[A_n\cos\left(\frac{n\pi}{4}t\right) + B_n \sin\left(\frac{n\pi}{4}t\right)\right]$

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A

b) For $A_n$:

$u(x,0) = f(x) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi}{2}x\right)\left[A_n\cos(0) + \cancel{B_n\sin(0)}^{0}\right]$

$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{2}x\right)$

MULTIPLY BOTH SIDES AND INTEGRATE:

$$\int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx = \frac{2}{2}A_n$$

$$\therefore A_n = \int_0^2 f(x)\sin\left(\frac{n\pi}{2}x\right)dx$$

For $B_n$:

$$u_t(x,0) = g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}x\right)\left[\cancel{A_n\left(-\frac{n\pi}{4}\sin\left(\frac{n\pi}{4}t\right)\right)} + B_n\left(\frac{n\pi}{4}\cos\left(\frac{n\pi}{4}t\right)\right)\right]$$

$$g(x) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{2}x\right)\left(B_n\frac{n\pi}{4}\right)$$

MULTIPLY BOTH SIDES AND INTEGRATE:

$$\int_0^2 g(x)\sin\left(\frac{n\pi}{2}x\right)dx = \frac{2}{2}B_n\frac{n\pi}{4}$$

$$\therefore B_n = \frac{4}{n\pi}\int_0^2 g(x)\sin\left(\frac{n\pi}{2}x\right)dx$$

c) IF $g(x) = 0 \Rightarrow B_n = 0$

IF $f(x) = 100$

$$A_n = \int_0^2 (100)\sin\left(\frac{n\pi}{2}x\right)dx = 100\left[-\left(\frac{2}{n\pi}\right)\cos\frac{n\pi}{2}x\right]_0^2$$

$$A_n = \frac{-200}{n\pi}\left(\cos(n\pi) - 1\right) \qquad n = 1, 2, 3, \dots$$

$$A_n = \begin{cases} \frac{400}{n\pi} & \text{IF } n \text{ IS ODD} \\ 0 & \text{IF } n \text{ IS EVEN} \end{cases}$$

**3. (CH3. Finite String: Fourier Series Solution)**

A (a) [50%] Display the series formula, **complete with derivation details**, for the solution $u(x,t)$ of the finite string problem

$$\begin{cases} u_{tt}(x,t) &= \frac{1}{4}u_{xx}(x,t), & 0<x<2, & t>0, \\ u(0,t) &= 0, & & t>0, \\ u(2,t) &= 0, & & t>0, \\ u(x,0) &= f(x), & 0<x<2, & \\ u_t(x,0) &= g(x), & 0<x<2. & \end{cases}$$

Symbols $f$ and $g$ should not appear explicitly in the series for $u(x,t)$. Expected in the formula for $u(x,t)$ are product solutions times constants.

A (b) [25%] Display explicit formulas for the Fourier coefficients which contains the symbols $f(x)$, $g(x)$.

A (c) [25%] Evaluate the Fourier coefficients when $f(x) = 100$ and $g(x) = 0$.

a. $u = X(x)T(t)$

$u_{tt} = \frac{1}{4} u_{xx}$

$T''X = \frac{1}{4} X''T$

$$\frac{T''}{\frac{1}{4}T} = \frac{X''}{X} = -\lambda$$

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(2) = 0 \quad \text{(from boundary conditions)} \end{cases}$$

$X = c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x)$

$X(0) = 0 = C_1$

$X(2) = C_2 \sin(2\sqrt{\lambda}) = 0$

$2\sqrt{\lambda} = n\pi$

$\sqrt{\lambda} = n\pi/2$

$\lambda = (n\pi/2)^2$

$\Rightarrow X = c_1 \sin(n\pi x/2)$

$T'' + \frac{1}{4}\lambda T = 0$

$$\begin{cases} T'' + \frac{1}{4}\left(\frac{n\pi}{2}\right)^2 = 0 \\ T \neq 0 \end{cases}$$

Use this page to start your solution. Attach extra pages as needed, then staple.

$T = d_1 \sin\left(\frac{1}{2}\left(\frac{n\pi}{2}\right)t\right) + d_2 \cos\left(\frac{1}{2}\left(\frac{n\pi}{2}\right)t\right)$

$= d_1 \sin(n\pi t/4) + d_2 \cos(n\pi t/4)$

modes: $\sin(n\pi x/2)\cos(n\pi t/4)$, $\sin(n\pi x/2)\sin(n\pi t/4)$

$$\Rightarrow u(x,t) = \sum_1^\infty a_n \sin(n\pi x/2)\cos(n\pi t/4) + \sum_1^\infty b_n \sin(n\pi x/2)\cos(n\pi t/4)$$

3b. $u(x,0) = \sum_1^{\infty} a_n \sin(n\pi x/2)\cos(0) + \sum_1 b_n \cancel{\sin(n\pi x/2)\sin(0)}$

$f(x) = \sum a_n \sin(n\pi x/2)$

$\int_0^2 f(x)\sin(m\pi x/2)\,dx = \sum a_n \int_0^2 \sin(n\pi x/2)\sin(m\pi x/2)\,dx$

$= 0$ except $n=m$:

$= a_n \int_0^2 \sin^2(n\pi x/2)\,dx$

$= a_n$

$\Rightarrow a_n = \int_0^2 f(x)\sin(n\pi x/2)\,dx$

$u_t(x,0) = \sum_1^{\infty} a_n \sin(n\pi x/2)\overset{0}{\cancel{\sin(0)}}(n\pi/4) + \sum_1^{\infty} b_n \sin(n\pi x/2)\overset{1}{\cos(0)}(n\pi/4)$

$g(x) = \sum b_n \frac{n\pi}{4} \sin(n\pi x/2)$

$\int_0^2 g(x)\sin(m\pi x/2)\,dx = \sum \frac{n\pi}{4} b_n \int_0^2 \sin(n\pi x/2)\sin(m\pi x/2)\,dx$

$= \frac{n\pi}{4} b_n \int_0^2 \sin^2(n\pi x/2)\,dx$

$= \frac{n\pi}{4} b_n$

$\Rightarrow b_n = \frac{4}{n\pi}\int_0^2 g(x)\sin(n\pi x/2)\,dx$

c. $f(x) = 100$:

$a_n = \int_0^2 100\sin(n\pi x/2)\,dx$

$= 100(-\cos(n\pi x/2))\cdot \frac{2}{n\pi}\Big|_0^2$

$= \frac{200}{n\pi}(-\cos(n\pi 2/2)+1)$

$a_n = \frac{200}{n\pi}(-(-1)^n+1)$ ✓

$g(x) = 0$:

$b_n = \frac{4}{n\pi}\int_0^2 0\,dx$

$b_n = 0$ ✓

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A (b) [25%] Display explicit formulas for the Fourier coefficients which contains the symbols $f(x)$, $g(x)$.

A (c) [25%] Evaluate the Fourier coefficients when $f(x) = 100$ and $g(x) = 0$.

a)

Let $U(x,t) = X(x)T(t)$ Then Plug into $U_{xx} = c^2 U_{tt}$

$\Rightarrow X''T = c^2 X T'' \Rightarrow \dfrac{X''}{X} = \dfrac{T''}{c^2 T} = k$ ✓

Then we get a system of eqn's

$X'' - (-k)X = 0$
$T'' - (-k)c^2 T = 0$ } Let $k<0$ b/c $k>0$ or $k=0$ yields Trivial solutions. Let $k = -\mu^2$

Solving for $X = C_1 \cos(\mu x) + C_2 \sin(\mu x)$

$X(0) = 0 \Rightarrow C_1 = 0$, $X(L) = 0 \Rightarrow \sin(\mu L) = 0 \Rightarrow \mu = \frac{n\pi}{L}$

Thus $X_n = \sin(\frac{n\pi}{L}x)$ Now subs $\mu$ into T eqn & let $\lambda_n = \frac{nc\pi}{L}$

Solving for T we get $T_n = c_1 \cos(\lambda_n t) + C_2 \sin(\lambda_n t)$

Then $U(x,t) = \sum_{n=1}^{\infty} X_n T_n = \sum_{n=1}^{\infty} \sin(\frac{n\pi}{L}x)(b_n \cos(\lambda_n t) + b_n^* \sin(\lambda_n t))$ ✓

Use this page to start your solution. Attach extra pages as needed, then staple.

Use $U(x,0) = f(x)$ to solve for $b_n$

$\Rightarrow \int_0^L f(x) \sin(\frac{n\pi}{L}x)\,dx = b_n \int_0^L \sin^2(\frac{n\pi}{L}x)\,dx = b_n \cdot \frac{L}{2}$

$\Rightarrow b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{n\pi}{L}x)\,dx$ ✓

Part (a) continued

Use $U_t(x,0) = g(x)$ to solve for $b_n^*$

$$U_t(x,0) = g(x) = \sum_{n=1}^{\infty} \lambda_n b_n^* \sin\left(\frac{n\pi}{L}x\right)$$

Then $\displaystyle\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx = \lambda_n b_n^* \int_0^L \sin^2\left(\frac{n\pi}{L}x\right) dx$

$$\Rightarrow b_n^* = \frac{2}{nc\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

) $\displaystyle b_n = \int_0^2 f(x) \sin\left(\frac{n\pi}{2}x\right) dx$

$$b_n^* = \frac{4}{n\pi} \int_0^2 g(x) \sin\left(\frac{n\pi}{2}x\right) dx$$

c) for $f(x) = 100$

$$b_n = -100 \cdot \frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}x\right)\Big|_0^2\right)$$

$$= \frac{200}{n\pi}\left(1-(-1)^n\right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{400}{n\pi} & \text{if } n \text{ is odd} \end{cases}$$

if $g(x) = 0$ then $b_n^* = 0$

**4. (CH3. Poisson Problem)**
Solve for $u(x,y)$ in the Poisson problem

$$\begin{cases} u_{xx}+u_{yy} = \sin(\pi x)\sin(2\pi y), & 0<x<1,\quad 0<y<1, \\ u(0,y) = 4\sin(3\pi y), & 0<y<1, \\ u(x,y) = 0 & \text{on the other 3 boundary edges.} \end{cases}$$

$a=b=1$

$\sum C_N \sinh\left(\frac{n\pi}{b}(a-x)\right)\sin\left(\frac{n\pi y}{b}\right)$

A

Product solution derivations are not expected. Expected solution details:

(1) Decomposition into two problems, $u = u_1 + u_2$. Draw figures.

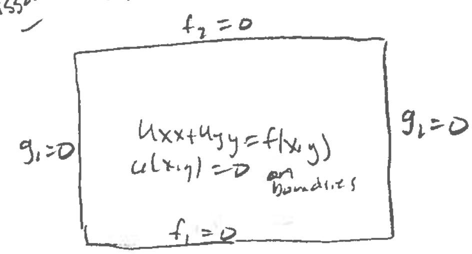
(2) Eigenfunctions $\phi_{mn}$ and eigenvalues $\lambda_{mn}$ of the Helmholtz equation $\phi_{xx}(x,y)+\phi_{yy}(x,y) = -\lambda\phi(x,y)$ with zero boundary conditions on the rectangle.

(3) Poisson problem defined for $u_1(x,y)$. Solution formula for $u_1(x,y)$. Fourier coefficient formula.

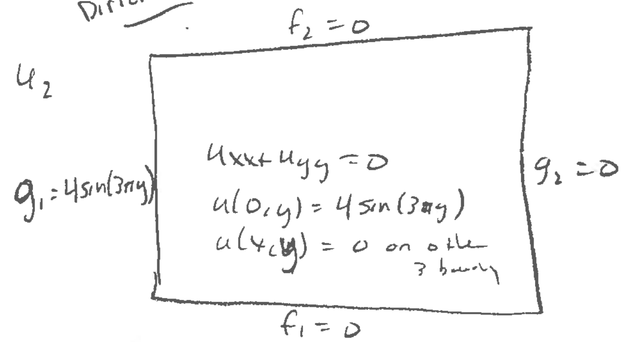
(4) Dirichlet problem defined for $u_2(x,y)$. Product solutions. Solution formula for $u_2(x,y)$. Fourier coefficient formula.

(5) Display the answer $u = u_1 + u_2$.

1) Poisson $u_1$:




Dirichlet $u_2$




2) $\phi_{mn} = \sin(n\pi x)\sin(m\pi y)$

$\lambda_{mn} = (m\pi)^2+(n\pi)^2$

3) $u_1$: $\begin{cases} u_{xx}+u_{yy} = \sin(\pi x)\sin(2\pi y) \\ u(x,y)=0 \quad \text{on boundries} \end{cases}$

$a=b=1$

$$\boxed{u_1(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} E_{mn}\sin(n\pi x)\sin(m\pi y)}$$

$$f(x,y) = \sin(\pi x)\sin(2\pi y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} -E_{mn}\lambda_{mn}\sin(n\pi x)\sin(m\pi y)$$

only non-zero for $n=1$ $m=2$

$$\int_0^1\int_0^1 \underbrace{\sin^2(\pi x)\sin^2(2\pi y)}_{(1/4)}\,dx\,dy = -E_{21}\lambda_{21}\int_0^1\int_0^1 \underbrace{\sin^2(\pi x)\sin^2(2\pi y)}_{(1/4)}\,dx\,dy$$

$$\frac{1}{4} = -E_{21}\lambda_{21}\frac{1}{4} \Rightarrow E_{21} = \frac{-1}{\lambda_{21}} \Rightarrow E_{21} = \frac{-1}{4\pi^2+\pi^2} = \boxed{\frac{-1}{5\pi^2}}$$

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4). $\begin{cases} u_{xx}+u_{yy}=0 \\ u(0,y)=4\sin(3\pi y) \\ u(x,y)=0 \quad \text{on other 3 borders} \end{cases}$

product Solutions

$\sinh\left(\frac{n\pi}{b}(a-x)\right)\sin\left(\frac{n\pi y}{b}\right)$

Solution formula

$$u_2(x,y)=\sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{b}(a-x)\right)\sin\left(\frac{n\pi y}{b}\right)$$

$a=b=1$

$$u_2(x,y)=\sum_{n=1}^{\infty} A_n \sinh(n\pi(1-x))\sin(n\pi y)$$

Fourier Coeffs

$$u(0,y)=4\sin(3\pi y)=\sum_{n=1}^{\infty} A_n \sinh(n\pi)\sin(n\pi y)$$

only valid when $n=3$

$$A_3=\int_0^1 4\sin^2(3\pi y)\,dy = A_3\sinh(3\pi)\int_0^1 \sin^2(3\pi y)\,dy$$

$\Rightarrow A_3\sinh(3\pi)=4$

$$\Rightarrow \boxed{A_3=\frac{4}{\sinh(3\pi)}}$$

5). $\boxed{u_1=\frac{-1}{5\pi^2}\sin(\pi x)\sin(2\pi y)+\frac{4\sinh(3\pi(1-x))\sin(3\pi y)}{\sinh(3\pi)}}$

A

**4. (CH3. Poisson Problem)**
Solve for $u(x, y)$ in the Poisson problem

$$\begin{cases} u_{xx} + u_{yy} &= \sin(\pi x)\sin(2\pi y), & 0 < x < 1, \quad 0 < y < 1, \\ u(0, y) &= 4\sin(3\pi y), & 0 < y < 1, \\ u(x, y) &= 0 & \text{on the other 3 boundary edges.} \end{cases}$$

Product solution derivations are not expected. Expected solution details:

(1) Decomposition into two problems, $u = u_1 + u_2$. Draw figures.

(2) Eigenfunctions $\phi_{mn}$ and eigenvalues $\lambda_{mn}$ of the Helmholtz equation $\phi_{xx}(x, y) + \phi_{yy}(x, y) = -\lambda\phi(x, y)$ with zero boundary conditions on the rectangle.

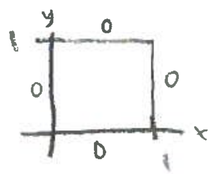
(3) Poisson problem defined for $u_1(x, y)$. Solution formula for $u_1(x, y)$. Fourier coefficient formula.

(4) Dirichlet problem defined for $u_2(x, y)$. Product solutions. Solution formula for $u_2(x, y)$. Fourier coefficient formula.

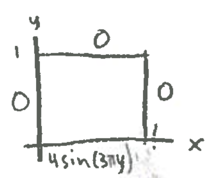
(5) Display the answer $u = u_1 + u_2$.

1. $u = u_1 + u_2$

$u_1$ : $\begin{cases} u_{xx} + u_{yy} = \sin(\pi x)\sin(2\pi y) & 0 < x < 1,\ 0 < y < 1 \\ u(x,y) = 0 \text{ on all edges:} \end{cases}$




$u_2$ : $\begin{cases} u_{xx} + u_{yy} = 0 \\ u(0,y) = 4\sin(3\pi y) \\ u(x,y) = 0 \text{ on other 3 edges} \end{cases}$




2. $u_1 = \sum\sum E_{mn}\phi_{mn} = \sum\sum E_{mn}\sin(m\pi x/a)\sin(n\pi y/b) = \sum\sum E_{mn}\sin(m\pi x)\sin(n\pi y)$

$a = b = 1$

$\phi_{mn}$ not defined

$u_{xx} = \sum\sum E_{mn}(m\pi)^2(-\sin(m\pi x))\sin(n\pi y)$

$u_{yy} = \sum\sum E_{mn}(n\pi)^2\sin(m\pi x)(-\sin(n\pi y))$

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$u_{xx} + u_{yy} = \sum\sum -E_{mn}((n\pi)^2 + (m\pi)^2)\sin(m\pi x)\sin(n\pi y)$

$\sin(\pi x)\sin(2\pi y) = \sum\sum -E_{mn}\lambda_{mn}\sin(m\pi x)\sin(n\pi y)$ where $\lambda_{mn} = (n\pi)^2 + (m\pi)^2$ ✓

by orthogonality, nonzero only when $m = 1$ and $n = 2$:

$\sin(\pi x)\sin(2\pi y) = -E_{12}\lambda_{12}\sin(\pi x)\sin(2\pi y)$

$\Rightarrow E_{12} = \frac{-1}{\lambda_{12}} = \frac{-1}{\pi^2 + (2\pi)^2} = \frac{-1}{5\pi^2} \Rightarrow u_1 = \frac{-1}{5\pi^2}\sin(\pi x)\sin(2\pi y)$ ✓

4. cont.

$u_2$ : modes : $\sin(n\pi y)\sinh(n\pi(1-x))$

$$u_2 = \sum a_n \sin(n\pi y)\sinh(n\pi(1-x))$$

$$u_2(0,y) = \sum a_n \sin(n\pi y)\sinh(n\pi)$$

$$4\sin(3\pi y) = \sum a_n \sin(n\pi y)\sinh(n\pi)$$

by orthogonality, nonzero only when $n=3$:

$$4\sin(3\pi y) = a_3 \sin(3\pi y)\sinh(3\pi)$$

$$a_3 = \frac{4}{\sinh(3\pi)}$$

$$\Rightarrow u_2 = \frac{4}{\sinh(3\pi)}\sin(3\pi y)\sinh(3\pi(1-x))$$

$$u = u_1 + u_2$$

$$u = \frac{-1}{5\pi^2}\sin(\pi x)\sin(2\pi y) + \frac{4}{\sinh(3\pi)}\sin(3\pi y)\sinh(3\pi(1-x))$$ ✓

Name. __ ______________ 3150 Final Exam S2013

**5.** **(CH7. Heat Equation and Gauss' Heat Kernel)**
Solve the insulated rod heat conduction problem

100

$$\begin{cases} u_t(x,t) &= \frac{1}{4}u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) &= f(x), & -\infty < x < \infty, \\ f(x) &= \begin{cases} 50 & 0 < x < 1, \\ 100 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

$c^2 = \frac{1}{4}$

$c = \frac{1}{2}$

$g_t(x) = \frac{2}{\sqrt{2t}} e^{-\frac{x^2}{t}}$

**Hint:** Use the heat kernel $g_t = \frac{1}{c\sqrt{2t}} e^{-\frac{x^2}{4c^2 t}}$, the error function $\mathbf{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$, and Fourier transform theory definitions to solve the problem. The answer is expressed in terms of the error function.

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(s) g_t(x-s)\,ds = \frac{1}{\sqrt{2\pi}}\left[\int_{-1}^{0} 100\, g_t(x-s)\,ds + \int_0^1 50\, g_t(x-s)\,ds\right]$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{2}{\sqrt{2t}}\left[100\int_{-1}^{0} e^{\frac{-(x-s)^2}{t}} ds + 50\int_0^1 e^{\frac{-(x-s)^2}{t}} ds\right]$$

CHANGE OF VARS

$z^2 = \frac{(x-s)^2}{t}$

$z = \frac{x-s}{\sqrt{t}}$

$dz = \frac{-ds}{\sqrt{t}}$

$\therefore ds = -\sqrt{t}\,dz$

$$= \frac{1}{\sqrt{\pi \cancel{t}}}\left[100\int_{z_1}^{z_2} -\cancel{\sqrt{t}}\, e^{-z^2} dz + 50\int_{z_2}^{z_3} -\cancel{\sqrt{t}}\, e^{-z^2} dz\right]$$

$$= \frac{1}{\sqrt{\pi}}\left[100\int_{z_2}^{z_1} e^{-z^2} dz + 50\int_{z_3}^{z_2} e^{-z^2} dz\right]$$

| $s$ | $z$ |
|---|---|
| $-1$ | $\frac{x+1}{\sqrt{2}} = z_1$ |
| $0$ | $\frac{x}{\sqrt{t}} = z_2$ |
| $1$ | $\frac{x-1}{\sqrt{t}} = z_3$ |

$$= \frac{1}{2}\left\{100\left[\mathrm{erf}(z_1) - \mathrm{erf}(z_2)\right] + 50\left[\mathrm{erf}(z_2) - \mathrm{erf}(z_3)\right]\right\}$$

$$= 50\,\mathrm{erf}(z_1) - 50\,\mathrm{erf}(z_2) + 25\,\mathrm{erf}(z_2) - 25\,\mathrm{ef}(z_3)$$

Use this page to start your solution. Attach extra pages as needed, then staple.

$$\boxed{u(x,t) = 50\,\mathrm{erf}(z_1) - 25\,\mathrm{erf}(z_2) - 25\,\mathrm{erf}(z_3) \quad \text{WHERE } z_1 = \frac{x+1}{\sqrt{2}},\ z_2 = \frac{x}{\sqrt{t}}\ \cancel{\&}\ z_3 = \frac{x-1}{\sqrt{t}}}$$

**5. (CH7. Heat Equation and Gauss' Heat Kernel)**
Solve the insulated rod heat conduction problem

$$\begin{cases} u_t(x,t) = \frac{1}{4}u_{xx}(x,t), & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = f(x), & -\infty < x < \infty, \\ f(x) = \begin{cases} 50 & 0 < x < 1, \\ 100 & -1 < x < 0 \\ 0 & \text{otherwise} \end{cases} \end{cases}$$

/ 100

**Hint:** Use the heat kernel $g_t = \frac{1}{c\sqrt{2t}}e^{-\frac{x^2}{4c^2t}}$, the error function $\mathbf{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-z^2}dz$, and Fourier transform theory definitions to solve the problem. The answer is expressed in terms of the error function.

Convoluting f(x) w/ the heat kernel we get

$$U(x,t) = \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(s)\, e^{-(x-s)^2/4c^2t}\, ds$$

$$= \frac{1}{\sqrt{\pi t}}\int_0^1 50\, e^{-(x-s)^2/t}\, ds \qquad z = \frac{x-s}{\sqrt{t}} \qquad dz = -\frac{1}{\sqrt{t}}$$

$$= \frac{-50}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{t}}}^{\frac{x-1}{\sqrt{t}}} e^{-z^2}\, dz$$

$$= 25\left(\mathrm{erf}\left(\frac{x}{\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x-1}{\sqrt{t}}\right)\right)$$

$$U_2(x,t) = \frac{1}{\sqrt{\pi t}}\int_{-1}^{0} 100\, e^{-(x-s)^2/t}\, ds \qquad u = \frac{x-s}{\sqrt{t}} \Rightarrow du = -\frac{1}{\sqrt{t}}$$

$$U_2(x,t) = \frac{100}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{t}}}^{\frac{x+1}{\sqrt{t}}} e^{-z^2}\, dz$$

$$= 50\left(\mathrm{erf}\left(\frac{x+1}{\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x}{\sqrt{t}}\right)\right)$$

Use this page to start your solution. Attach extra pages as needed, then staple.

$$\Rightarrow U(x,t) = 50\left(\mathrm{erf}\left(\frac{x+1}{\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x}{\sqrt{t}}\right)\right) + 25\left(\mathrm{erf}\left(\frac{x}{\sqrt{t}}\right) - \mathrm{erf}\left(\frac{x-1}{\sqrt{t}}\right)\right)$$