4.6 Orthogonal Vectors in $\mathbf{R}^n$

In this section we show that the geometrical concepts of *distance* and *angle* in n-dimensional space can be based on the definition of the *dot product* of two vectors in $\mathbf{R}^n$. Recall from elementary calculus that the dot product $\mathbf{u} \cdot \mathbf{v}$ of two 3-dimensional vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is (by definition) the sum

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

of the products of corresponding scalar components of the two vectors.

Similarly, the **dot product** $\mathbf{u} \cdot \mathbf{v}$ of two n-dimensional vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is defined by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n \tag{1}$$

(with one additional scalar product term for each additional dimension). And just as in $\mathbf{R}^3$, it follows readily from the formula in (1) that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors in $\mathbf{R}^n$

and c is a scalar, then

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} \qquad \text{(symmetry)} \tag{2}$$

$$\mathbf{u}\cdot(\mathbf{v}+\mathbf{w}) = \mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w} \qquad \text{(distributivity)} \tag{3}$$

$$(c\mathbf{u})\cdot\mathbf{v} = c(\mathbf{u}\cdot\mathbf{v}) \qquad \text{(homogeneity)} \tag{4}$$

$$\left.\begin{array}{l} \mathbf{u}\cdot\mathbf{u} \geqq 0; \\ \mathbf{u}\cdot\mathbf{u} = 0 \quad \text{if and only if} \\ \mathbf{u} = \mathbf{0}. \end{array}\right\} \qquad \text{(positivity)} \tag{5}$$

Therefore, the dot product in $\mathbf{R}^n$ is an example of an *inner product*.

DEFINITION Inner Product

An **inner product** on a vector space V is a function that associates with each pair of vectors $\mathbf{u}$ and $\mathbf{v}$ in V a scalar $\langle\mathbf{u},\mathbf{v}\rangle$ such that, if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are vectors and c is a scalar, then

(i) $\langle\mathbf{u},\mathbf{v}\rangle = \langle\mathbf{v},\mathbf{u}\rangle$;

(ii) $\langle\mathbf{u},\mathbf{v}+\mathbf{w}\rangle = \langle\mathbf{u},\mathbf{v}\rangle + \langle\mathbf{u},\mathbf{w}\rangle$;

(iii) $\langle c\mathbf{u},\mathbf{v}\rangle = c\langle\mathbf{u},\mathbf{v}\rangle$;

(iv) $\langle\mathbf{u},\mathbf{u}\rangle \geqq 0$; $\langle\mathbf{u},\mathbf{u}\rangle = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

The dot product on $\mathbf{R}^n$ is sometimes called the **Euclidean inner product**, and with this inner product $\mathbf{R}^n$ is sometimes called **Euclidean *n*-dimensional space**. We can use any of the notations in

$$\mathbf{u}\cdot\mathbf{v} = \langle\mathbf{u},\mathbf{v}\rangle = \mathbf{u}^T\mathbf{v}$$

for the dot product of the two $n \times 1$ column vectors $\mathbf{u}$ and $\mathbf{v}$. (Note in the last expression that $\mathbf{u}^T = (u_1, u_2, \ldots, u_n)^T$ is the $1 \times n$ *row* vector with the indicated entries, so the 1×1 matrix product $\mathbf{u}^T\mathbf{v}$ is simply a scalar.) Here we will ordinarily use the notation $\mathbf{u}\cdot\mathbf{v}$.

The **length** $|\mathbf{u}|$ of the vector $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ is defined as follows:

$$|\mathbf{u}| = \sqrt{(\mathbf{u}\cdot\mathbf{u})} = \left(u_1^2 + u_2^2 + \cdots + u_n^2\right)^{1/2}. \tag{6}$$

Note that the case $n = 2$ is a consequence of the familiar Pythagorean formula in the plane.

Theorem 1 gives one of the most important inequalities in mathematics. Many proofs are known, but none of them seems direct and well motivated.

THEOREM 1 The Cauchy-Schwarz Inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^n$, then

$$|\mathbf{u}\cdot\mathbf{v}| \leqq |\mathbf{u}||\mathbf{v}|. \tag{7}$$

***Proof*:** If $\mathbf{u} = \mathbf{0}$, then $|\mathbf{u} \cdot \mathbf{v}| = |\mathbf{u}| = 0$, so the inequality is satisfied trivially. If $\mathbf{u} \neq \mathbf{0}$, then we let $a = \mathbf{u} \cdot \mathbf{u}$, $b = 2\mathbf{u} \cdot \mathbf{v}$, and $c = \mathbf{v} \cdot \mathbf{v}$. For any real number x, the distributivity and positivity properties of the dot product then yield

$$\begin{aligned} 0 &\leq (x\mathbf{u} + \mathbf{v}) \cdot (x\mathbf{u} + \mathbf{v}) \\ &= (\mathbf{u} \cdot \mathbf{u})x^2 + 2(\mathbf{u} \cdot \mathbf{v})x + (\mathbf{v} \cdot \mathbf{v}), \end{aligned}$$

so that

$$0 \leq ax^2 + bx + c.$$

Thus the quadratic equation $ax^2 + bx + c = 0$ either has no real roots or has a repeated real root. Hence the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

implies that the discriminant $b^2 - 4ac$ cannot be positive; that is, $b^2 \leq 4ac$, so

$$4(\mathbf{u} \cdot \mathbf{v})^2 \leq 4(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}).$$

We get the Cauchy-Schwarz inequality in (7) when we take square roots, remembering that the numbers $|\mathbf{u}| = (\mathbf{u} \cdot \mathbf{u})^{1/2}$ and $|\mathbf{v}| = (\mathbf{v} \cdot \mathbf{v})^{1/2}$ are nonnegative. ■

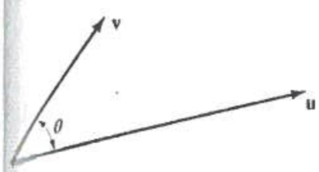

FIGURE 4.6.1. The angle θ between the vectors **u** and **v**.

The Cauchy-Schwarz inequality enables us to *define* the angle θ between the nonzero vectors **u** and **v**. (See Figure 4.6.1.) Division by the positive number $|\mathbf{u}||\mathbf{v}|$ in (7) yields $|\mathbf{u} \cdot \mathbf{v}|/(|\mathbf{u}||\mathbf{v}|) \leq 1$, so

$$-1 \leq \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} \leq +1. \tag{8}$$

Hence there is a unique angle θ between 0 and π radians, inclusive (that is, between 0° and 180°), such that

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|}. \tag{9}$$

Thus we obtain the same geometric interpretation

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta \tag{10}$$

of the dot product in $\mathbf{R}^n$ as one sees (for 3-dimensional vectors) in elementary calculus textbooks—for instance, see Section 11.2 of Edwards and Penney, *Calculus: Early Transcendentals*, 7th edition (Prentice Hall) 2008.

On the basis of (10) we call the vectors **u** and **v** **orthogonal** provided that

$$\mathbf{u} \cdot \mathbf{v} = 0. \tag{11}$$

If **u** and **v** are nonzero vectors this means that $\cos\theta = 0$, so $\theta = \pi/2$ (90°). Note that $\mathbf{u} = \mathbf{0}$ satisfies (11) for all **v**, so the zero vector is orthogonal to *every* vector.

Example 1 Find the angle θ_n in $\mathbf{R}^n$ between the x_1-axis and the line through the origin and the point $(1, 1, \ldots, 1)$.

Solution We take $\mathbf{u} = (1, 0, 0, \ldots, 0)$ on the x_1-axis and $\mathbf{v} = (1, 1, \ldots, 1)$. Then $|\mathbf{u}| = 1$, $|\mathbf{v}| = \sqrt{n}$, and $\mathbf{u} \cdot \mathbf{v} = 1$, so the formula in (9) gives

$$\cos\theta_n = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{1}{\sqrt{n}}.$$

For instance, if

$$\begin{aligned}
n &= 3, &&\text{then} & \theta_3 &= \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 0.9553 \quad (55^\circ);\\
n &= 4, &&\text{then} & \theta_4 &= \cos^{-1}\left(\frac{1}{\sqrt{4}}\right) \approx 1.0472 \quad (60^\circ);\\
n &= 5, &&\text{then} & \theta_5 &= \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \approx 1.1071 \quad (63^\circ);\\
n &= 100, &&\text{then} & \theta_{100} &= \cos^{-1}\left(\frac{1}{10}\right) \approx 1.4706 \quad (84^\circ).
\end{aligned}$$

It is interesting to note that θ_n increases as n increases. Indeed, θ_n approaches $\cos^{-1}(0) = \pi/2$ (90°) as n increases without bound (so that $1/\sqrt{n}$ approaches zero). ■

In addition to angles, the dot product provides a definition of distance in $\mathbf{R}^n$. The **distance** $d(\mathbf{u}, \mathbf{v})$ between the points (vectors) $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ is defined to be

$$\begin{aligned}
d(\mathbf{u}, \mathbf{v}) &= |\mathbf{u} - \mathbf{v}|\\
&= \left[(u_1 - v_1)^2 + (u_2 - v_2)^2 + \cdots + (u_n - v_n)^2\right]^{1/2}.
\end{aligned} \tag{12}$$

Example 2 The distance between the points $\mathbf{u} = (1, -1, -2, 3, 5)$ and $\mathbf{v} = (4, 3, 4, 5, 9)$ in $\mathbf{R}^5$ is

$$|\mathbf{u} - \mathbf{v}| = \sqrt{3^2 + 4^2 + 6^2 + 2^2 + 4^2} = \sqrt{81} = 9.$$ ■

The *triangle inequality* of Theorem 2 relates the three sides of the triangle shown in Figure 4.6.2.

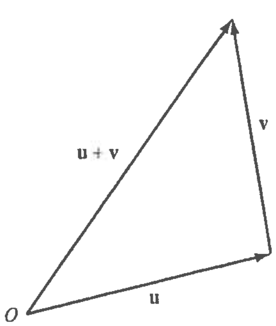

FIGURE 4.6.2. The "triangle" of the triangle inequality.

THEOREM 2 The Triangle Inequality

If $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbf{R}^n$, then

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|. \tag{13}$$

Proof: We apply the Cauchy-Schwarz inequality to find that

$$\begin{aligned}
|\mathbf{u} + \mathbf{v}|^2 &= (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})\\
&= \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v}\\
&\leq \mathbf{u} \cdot \mathbf{u} + 2|\mathbf{u}||\mathbf{v}| + \mathbf{v} \cdot \mathbf{v}\\
&= |\mathbf{u}|^2 + 2|\mathbf{u}||\mathbf{v}| + |\mathbf{v}|^2,
\end{aligned} \tag{14}$$

and therefore

$$|\mathbf{u}+\mathbf{v}|^2 \leqq (|\mathbf{u}|+|\mathbf{v}|)^2.$$

We now get (13) when we take square roots. ■

The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if $\mathbf{u}\cdot\mathbf{v}=0$, so line (14) in the proof of the triangle inequality yields the fact that the **Pythagorean formula**

$$|\mathbf{u}+\mathbf{v}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 \tag{15}$$

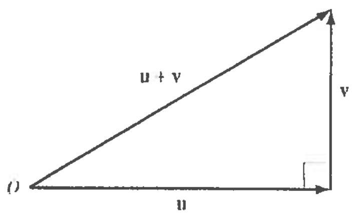

FIGURE 4.6.3. A right triangle in $\mathbf{R}^n$.

holds if and only if the triangle with "adjacent side vectors" $\mathbf{u}$ and $\mathbf{v}$ is a right triangle with hypotenuse vector $\mathbf{u}+\mathbf{v}$ (see Figure 4.6.3).

The following theorem states a simple relationship between orthogonality and linear independence.

THEOREM 3 Orthogonality and Linear Independence

If the nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are mutually orthogonal—that is, each two of them are orthogonal—then they are linearly independent.

***Proof*:** Suppose that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0},$$

where, as usual, $c_1, c_2, \ldots, c_k$ are scalars. When we take the dot product of each side of this equation with $\mathbf{v}_i$, we find that

$$c_i\mathbf{v}_i \cdot \mathbf{v}_i = c_i|\mathbf{v}_i|^2 = 0.$$

Now $|\mathbf{v}_i| \neq 0$ because $\mathbf{v}_i$ is a nonzero vector. It follows that $c_i = 0$. Thus $c_1 = c_2 = \cdots = c_k = 0$, and therefore the mutually orthogonal nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent. ■

In particular, any set of n mutually orthogonal nonzero vectors in $\mathbf{R}^n$ constitutes a basis for $\mathbf{R}^n$. Such a basis is called an **orthogonal basis**. For instance, the standard unit vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ form an orthogonal basis for $\mathbf{R}^n$.

Orthogonal Complements

Now we want to relate orthogonality to the solution of systems of linear equations. Consider the homogeneous linear system

$$\mathbf{Ax} = \mathbf{0} \tag{16}$$

of m equations in n unknowns. If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are the row vectors of the $m \times n$ coefficient matrix $\mathbf{A}$, then the system looks like

$$\begin{bmatrix} \text{———} \mathbf{v}_1 \text{———} \\ \text{———} \mathbf{v}_2 \text{———} \\ \vdots \\ \text{———} \mathbf{v}_m \text{———} \end{bmatrix} \begin{bmatrix} | \\ \mathbf{x} \\ | \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1\cdot\mathbf{x} \\ \mathbf{v}_2\cdot\mathbf{x} \\ \vdots \\ \mathbf{v}_m\cdot\mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Consequently, it is clear that $\mathbf{x}$ is a solution vector of $\mathbf{Ax} = \mathbf{0}$ if and only if $\mathbf{x}$ is orthogonal to each row vector of $\mathbf{A}$. But in the latter event $\mathbf{x}$ is orthogonal to every linear combination of row vectors of $\mathbf{A}$ because

$$\begin{aligned}\mathbf{x} \cdot (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m) \\ &= c_1\mathbf{x} \cdot \mathbf{v}_1 + c_2\mathbf{x} \cdot \mathbf{v}_2 + \cdots + c_m\mathbf{x} \cdot \mathbf{v}_m \\ &= (c_1)(0) + (c_2)(0) + \cdots + (c_m)(0) = 0.\end{aligned}$$

Thus we have shown that *the vector* $\mathbf{x}$ *in* $\mathbf{R}^n$ *is a solution vector of* $\mathbf{Ax} = \mathbf{0}$ *if and only if* $\mathbf{x}$ *is orthogonal to each vector in the row space* Row($\mathbf{A}$) *of the matrix* $\mathbf{A}$. This situation motivates the following definition.

DEFINITION The Orthogonal Complement of a Subspace

The vector $\mathbf{u}$ is **orthogonal** to the subspace V of $\mathbf{R}^n$ provided that $\mathbf{u}$ is orthogonal to every vector in V. The **orthogonal complement** $V^\perp$ (read "V perp") of V is the set of all those vectors in $\mathbf{R}^n$ that are orthogonal to the subspace V.

If $\mathbf{u}_1$ and $\mathbf{u}_2$ are vectors in $V^\perp$, $\mathbf{v}$ is in V, and c_1 and c_2 are scalars, then

$$\begin{aligned}(c_1\mathbf{u}_1 + c_2\mathbf{u}_2) \cdot \mathbf{v} &= c_1\mathbf{u}_1 \cdot \mathbf{v} + c_2\mathbf{u}_2 \cdot \mathbf{v} \\ &= (c_1)(0) + (c_2)(0) = 0.\end{aligned}$$

Thus any linear combination of vectors in $V^\perp$ is orthogonal to every vector in V and hence is a vector in $V^\perp$. Therefore *the orthogonal complement* $V^\perp$ *of a subspace* V *is itself a subspace of* $\mathbf{R}^n$. The standard picture of two complementary subspaces V and $V^\perp$ consists of an orthogonal line and plane through the origin in $\mathbf{R}^3$ (see Fig. 4.6.4). The proofs of the remaining parts of Theorem 4 are left to the problems.

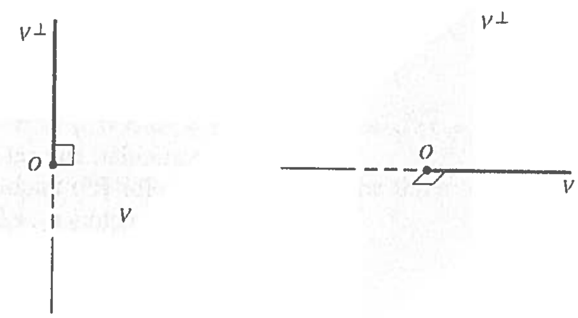

FIGURE 4.6.4. Orthogonal complements.

THEOREM 4 Properties of Orthogonal Complements

Let V be a subspace of $\mathbf{R}^n$. Then

1. Its orthogonal complement $V^\perp$ is also a subspace of $\mathbf{R}^n$;
2. The only vector that lies in both V and $V^\perp$ is the zero vector;
3. The orthogonal complement of $V^\perp$ is V—that is, $(V^\perp)^\perp = V$;
4. If S is a spanning set for V, then the vector $\mathbf{u}$ is in $V^\perp$ if and only if $\mathbf{u}$ is orthogonal to every vector in S.

In our discussion of the homogeneous linear system $\mathbf{Ax} = \mathbf{0}$ in (16), we showed that a vector space lies in the **null space** Null(**A**) of **A**—that is, in the solution space of $\mathbf{Ax} = \mathbf{0}$—if and only if it is orthogonal to each vector in the row space of **A**. In the language of orthogonal complements, this proves Theorem 5.

THEOREM 5 The Row Space and the Null Space

Let **A** be an $m \times n$ matrix. Then the row space Row(**A**) and the null space Null(**A**) are orthogonal complements in $\mathbf{R}^n$. That is,

$$\text{If} \quad V = \text{Row}(\mathbf{A}), \quad \text{then} \quad V^\perp = \text{Null}(\mathbf{A}). \tag{17}$$

Now suppose that a subspace V of $\mathbf{R}^n$ is given, with $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ a set of vectors that span V. For instance, these vectors may form a given basis for V. Then the implication in (17) provides the following algorithm for finding a basis for the orthogonal complement $V^\perp$ of V.

1. Let **A** be the $m \times n$ matrix with row vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$.
2. Reduce **A** to echelon form and use the algorithm of Section 4.4 to find a basis $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ for the solution space Null(**A**) of $\mathbf{Ax} = \mathbf{0}$. Because $V^\perp =$ Null(**A**), this will be a basis for the orthogonal complement of V.

Example 3 Let V be the 1-dimensional subspace of $\mathbf{R}^3$ spanned by the vector $\mathbf{v}_1 = (1, -3, 5)$. Then

$$\mathbf{A} = \begin{bmatrix} 1 & -3 & 5 \end{bmatrix}$$

and our linear system $\mathbf{Ax} = \mathbf{0}$ consists of the single equation

$$x_1 - 3x_2 + 5x_3 = 0.$$

If $x_2 = s$ and $x_3 = t$, then $x_1 = 3s - 5t$. With $s = 1$ and $t = 0$ we get the solution vector $\mathbf{u}_1 = (3, 1, 0)$, whereas with $s = 0$ and $t = 1$ we get the solution vector $\mathbf{u}_2 = (-5, 0, 1)$. Thus the orthogonal complement $V^\perp$ is the 2-dimensional subspace of $\mathbf{R}^3$ having $\mathbf{u}_1 = (3, 1, 0)$ and $\mathbf{u}_2 = (-5, 0, 1)$ as basis vectors. ■

Example 4 Let V be the 2-dimensional subspace of $\mathbf{R}^5$ that has $\mathbf{v}_1 = (1, 2, 1, -3, -3)$ and $\mathbf{v}_2 = (2, 5, 6, -10, -12)$ as basis vectors. The matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & -3 & -3 \\ 2 & 5 & 6 & -10 & -12 \end{bmatrix}$$

with row vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ has reduced echelon form

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & -7 & 5 & 9 \\ 0 & 1 & 4 & -4 & -6 \end{bmatrix}.$$

Hence the solution space of $\mathbf{Ax} = \mathbf{0}$ is described parametrically by

$$x_3 = r, \quad x_4 = s, \quad x_5 = t,$$
$$x_2 = -4r + 4s + 6t$$
$$x_1 = 7r - 5s - 9t.$$

Then the choice

$$\begin{aligned} r = 1, \quad s = 0, \quad t = 0 \quad &\text{yields} \quad \mathbf{u}_1 = (7, -4, 1, 0, 0); \\ r = 0, \quad s = 1, \quad t = 0 \quad &\text{yields} \quad \mathbf{u}_2 = (-5, 4, 0, 1, 0); \\ r = 0, \quad s = 0, \quad t = 1 \quad &\text{yields} \quad \mathbf{u}_3 = (-9, 6, 0, 0, 1). \end{aligned}$$

Thus the orthogonal complement $V^\perp$ is the 3-dimensional subspace of $\mathbf{R}^5$ with basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$. ■

Observe that $\dim V + \dim V^\perp = 3$ in Example 3, but $\dim V + \dim V^\perp = 5$ in Example 4. It is no coincidence that in each case the dimensions of V and $V^\perp$ add up to the dimension n of the Euclidean space containing them. To see why, suppose that V is a subspace of $\mathbf{R}^n$ and let $\mathbf{A}$ be an $m \times n$ matrix whose row vectors span V. Then Equation (12) in Section 4.5 implies that

$$\operatorname{rank}(\mathbf{A}) + \dim \operatorname{Null}(\mathbf{A}) = n.$$

But

$$\dim V = \dim \operatorname{Row}(\mathbf{A}) = \operatorname{rank}(\mathbf{A})$$

and

$$\dim V^\perp = \dim \operatorname{Null}(\mathbf{A})$$

by Theorem 5, so it follows that

$$\dim V + \dim V^\perp = n. \tag{18}$$

Moreover, it should be apparent intuitively that if

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\} \text{ is a basis for } V$$

and

$$\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} \text{ is a basis for } V^\perp,$$

then

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m, \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\} \text{ is a basis for } \mathbf{R}^n.$$

That is, *the union of a basis for V and a basis for $V^\perp$ is a basis for $\mathbf{R}^n$*. In Problem 34 of this section we ask you to prove that this is so.

4.6 Problems

In Problems 1–4, determine whether the given vectors are mutually orthogonal.

1. $\mathbf{v}_1 = (2, 1, 2, 1)$, $\mathbf{v}_2 = (3, -6, 1, -2)$, $\mathbf{v}_3 = (3, -1, -5, 5)$

2. $\mathbf{v}_1 = (3, -2, 3, -4)$, $\mathbf{v}_2 = (6, 3, 4, 6)$, $\mathbf{v}_3 = (17, -12, -21, 3)$

3. $\mathbf{v}_1 = (5, 2, -4, -1)$, $\mathbf{v}_2 = (3, -5, 1, 1)$, $\mathbf{v}_3 = (3, 0, 8, -17)$

4. $\mathbf{v}_1 = (1, 2, 3, -2, 1)$, $\mathbf{v}_2 = (3, 2, 3, 6, -4)$, $\mathbf{v}_3 = (6, 2, -4, 1, 4)$

In Problems 5–8, the three vertices A, B, and C of a triangle are given. Prove that each triangle is a right triangle by showing that its sides a, b, and c satisfy the Pythagorean relation $a^2 + b^2 = c^2$.

5. $A(6, 6, 5, 8)$, $B(6, 8, 6, 5)$, $C(5, 7, 4, 6)$

6. $A(3, 5, 1, 3)$, $B(4, 2, 6, 4)$, $C(1, 3, 4, 2)$

7. $A(4, 5, 3, 5, -1)$, $B(3, 4, -1, 4, 4)$, $C(1, 3, 1, 3, 1)$

8. $A(2, 8, -3, -1, 2)$, $B(-2, 5, 6, 2, 12)$, $C(-5, 3, 2, -3, 5)$

9–12. Find the acute angles (in degrees) of each of the right triangles of Problems 5–8, respectively.