

Chapter 7

Higher Dimensional Partial Differential Equations

7.1 Introduction

In our discussion of partial differential equations, we have solved many problems by the method of separation of variables, but all involved only two independent variables:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\begin{aligned} \frac{\partial u}{\partial t} &= k \frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial t^2} &= c^2 \frac{\partial^2 u}{\partial x^2} \\ c\rho \frac{\partial u}{\partial t} &= \frac{\partial}{\partial x} \left(K_0 \frac{\partial u}{\partial x} \right) & \rho \frac{\partial^2 u}{\partial t^2} &= T_0 \frac{\partial^2 u}{\partial x^2}. \end{aligned}$$

In this chapter we show how to extend the method of separation of variables to problems with more than two independent variables.

In particular, we discuss techniques to analyze the heat equation (with constant thermal properties) in two and three dimensions,

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (\text{two dimensions}) \quad (7.1.1)$$

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \quad (\text{three dimensions}) \quad (7.1.2)$$

for various physical regions with various boundary conditions. Also of interest will be the steady-state heat equation, Laplace's equation, in three dimensions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

In all these problems, the partial differential equation has at least three independent variables. Other physical problems, not related to the flow of thermal energy, may also involve more than two independent variables. For example, the vertical displacement u of a vibrating membrane satisfies the two-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

It should also be mentioned that in acoustics, the perturbed pressure u satisfies the three-dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right).$$

We will discuss and analyze some of these problems.

7.2 Separation of the Time Variable

We will show that similar methods can be applied to a variety of problems. We will begin by discussing the vibrations of a membrane of any shape, and follow that with some analysis for the conduction of heat in any two- or three-dimensional region.

7.2.1 Vibrating Membrane: Any Shape

Let us consider the displacement u of a vibrating membrane of any shape. Later (Secs. 7.3 and 7.7) we will specialize our result to rectangular and circular membranes. The displacement $u(x, y, t)$ satisfies the two-dimensional wave equation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)}. \quad (7.2.1)$$

The initial conditions will be

$$u(x, y, 0) = \alpha(x, y) \quad (7.2.2)$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y), \quad (7.2.3)$$

but as usual they will be ignored at first when separating variables. A homogeneous boundary condition will be given along the entire boundary; $u = 0$ on the boundary is the most common condition. However, it is possible, for example, for the displacement to be zero on only part of the boundary and for the rest of the boundary to be "free." There are many other possible boundary conditions.

Let us now apply the method of separation of variables. We begin by showing that the time variable can be separated out from the problem for a membrane of

any shape by seeking product solutions of the following form:

$$u(x, y, t) = h(t)\phi(x, y). \quad (7.2.4)$$

Here $\phi(x, y)$ is an as yet unknown function of the two variables x and y . We do not (at this time) specify further $\phi(x, y)$ since we might expect different results in different geometries or with different boundary conditions. Later, we will show that for rectangular membranes $\phi(x, y) = F(x)G(y)$, while for circular membranes $\phi(x, y) = F(r)G(\theta)$; that is, the form of further separation depends on the geometry. It is for this reason that we begin by analyzing the general form (7.2.4). In fact, for most regions that are *not* geometrically as simple as rectangles and circles, $\phi(x, y)$ cannot be separated further. If (7.2.4) is substituted into the equation for a vibrating membrane, (7.2.1), then the result is

$$\phi(x, y) \frac{d^2 h}{dt^2} = c^2 h(t) \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right). \quad (7.2.5)$$

We will attempt to proceed as we did when there were only two independent variables. Time can be separated from (7.2.5) by dividing by $h(t)\phi(x, y)$ (and an additional division by the constant c^2 is convenient):

$$\frac{1}{c^2} \frac{1}{h} \frac{d^2 h}{dt^2} = \frac{1}{\phi} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) = -\lambda. \quad (7.2.6)$$

The left-hand side of the first equation is only a function of time, while the right-hand side is only a function of space (x and y). Thus, the two (as before) must equal a separation constant. Again, we must decide what notation is convenient for the separation constant, λ or $-\lambda$. A quick glance at the resulting ordinary differential equation for $h(t)$ shows that $-\lambda$ is more convenient (as will be explained). We thus obtain two equations, but unlike the case of two independent variables, one of the equations is itself still a partial differential equation:

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \quad (7.2.7)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi. \quad (7.2.8)$$

The notation $-\lambda$ for the separation constant was chosen because the time-dependent differential equation (7.2.7) has oscillatory solutions if $\lambda > 0$. If $\lambda > 0$, then h is a linear combination of $\sin c\sqrt{\lambda}t$ and $\cos c\sqrt{\lambda}t$; it oscillates with frequency $c\sqrt{\lambda}$. The values of λ determine the natural frequencies of oscillation of a vibrating

membrane. However, we are not guaranteed that $\lambda > 0$. To show that $\lambda > 0$, we must analyze the resulting eigenvalue problem, (7.2.8), where ϕ is subject to a homogeneous boundary condition along the entire boundary (e.g., $\phi = 0$ on the boundary). Here the eigenvalue problem itself involves a linear homogeneous partial differential equation. Shortly, we will show that $\lambda > 0$ by introducing a Rayleigh quotient applicable to (7.2.8). Before analyzing (7.2.8), we will show that it arises in other contexts.

7.2.2 Heat Conduction: Any Region

We will analyze the flow of thermal energy in any two-dimensional region. We begin by seeking product solutions of the form

$$u(x, y, t) = h(t)\phi(x, y) \quad (7.2.9)$$

for the two-dimensional heat equation, assuming constant thermal properties and no sources, (7.1.1). By substituting (7.2.9) into (7.1.1) and after dividing by $kh(t)\phi(x, y)$, we obtain

$$\frac{1}{h} \frac{dh}{dt} = \frac{1}{\phi} \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right). \quad (7.2.10)$$

A separation constant in the form $-\lambda$ is introduced so that the time-dependent part of the product solution exponentially decays (if $\lambda > 0$) as expected, rather than exponentially grows. Then, the two equations are

$$\begin{aligned} \frac{dh}{dt} &= -\lambda kh \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= -\lambda \phi. \end{aligned} \quad (7.2.11)$$

The eigenvalue λ relates to the decay rate of the time-dependent part. The eigenvalue λ is determined by the boundary value problem, again consisting of the partial differential equation (7.2.11) with a corresponding boundary condition on the entire boundary of the region.

For heat flow in any three-dimensional region, (7.1.2) is valid. A product solution,

$$u(x, y, z, t) = h(t)\phi(x, y, z), \quad (7.2.12)$$

may still be sought, and after separating variables, we obtain equations similar to

(7.2.11),

$$\boxed{\begin{aligned} \frac{dh}{dt} &= -\lambda kh \\ \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} &= -\lambda \phi. \end{aligned}} \quad (7.2.13)$$

The eigenvalue λ is determined by finding those values of λ for which nontrivial solutions of (7.2.13) exist, subject to a homogeneous boundary condition on the entire boundary.

7.2.3 Summary

In situations described in this section the spatial part $\phi(x, y)$ or $\phi(x, y, z)$ of the solution of the partial differential equation satisfies the eigenvalue problem consisting of the partial differential equation,

$$\boxed{\nabla^2 \phi = -\lambda \phi,} \quad (7.2.14)$$

with ϕ satisfying appropriate homogeneous boundary conditions, which may be of the form [see (1.5.2) and (4.5.5)]

$$\alpha \phi + \beta \nabla \phi \cdot \hat{n} = 0, \quad (7.2.15)$$

where α and β can depend on x, y , and z . If $\beta = 0$, (7.2.15) is the fixed boundary condition. If $\alpha = 0$, (7.2.15) is the insulated or free boundary condition. If both $\alpha \neq 0$ and $\beta \neq 0$, then (7.2.15) is the higher-dimensional version of Newton's law of cooling or the elastic boundary condition. In Sec. 7.4 we will describe general results for this two- or three-dimensional eigenvalue problem, similar to our theorems concerning the general one-dimensional Sturm-Liouville eigenvalue problem. However, first we will describe the solution of a simple two-dimensional eigenvalue problem in a situation in which $\phi(x, y)$ may be further separated, producing two one-dimensional eigenvalue problems.

EXERCISES 7.2

- 7.2.1. For a vibrating membrane of any shape that satisfies (7.2.1), show that (7.2.14) results after separating time.
- 7.2.2. For heat conduction in any two-dimensional region that satisfies (7.1.1), show that (7.2.14) results after separating time.
- 7.2.3. (a) Obtain product solutions, $\phi = f(x)g(y)$, of (7.2.14) that satisfy $\phi = 0$ on the four sides of a rectangle. (*Hint*: If necessary, see Sec. 7.3.)

- (b) Using part (a), solve the initial value problem for a vibrating rectangular membrane (fixed on all sides).
- (c) Using part (a), solve the initial value problem for the two-dimensional heat equation with zero temperature on all sides.

7.3 Vibrating Rectangular Membrane

In this section we analyze the vibrations of a rectangular membrane, as sketched in Fig. 7.3.1. The vertical displacement $u(x, y, t)$ of the membrane satisfies the two-dimensional wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \quad (7.3.1)$$

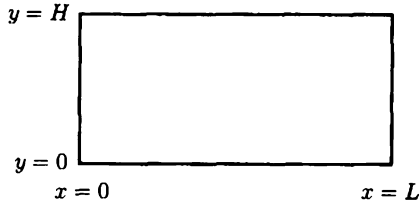


Figure 7.3.1 Rectangular membrane.

We suppose that the boundary is given such that all four sides are fixed with zero displacement:

$$u(0, y, t) = 0 \quad u(x, 0, t) = 0 \quad (7.3.2)$$

$$u(L, y, t) = 0 \quad u(x, H, t) = 0. \quad (7.3.3)$$

We ask what is the displacement of the membrane at time t if the initial position and velocity are given:

$$u(x, y, 0) = \alpha(x, y) \quad (7.3.4)$$

$$\frac{\partial u}{\partial t}(x, y, 0) = \beta(x, y). \quad (7.3.5)$$

As we indicated in Sec. 7.2.1, since the partial differential equation and the boundary conditions are linear and homogeneous, we apply the method of separation of variables. First, we separate only the time variable by seeking product solutions in the form

$$u(x, y, t) = h(t)\phi(x, y). \quad (7.3.6)$$

According to our earlier calculation, we are able to introduce a separation constant $-\lambda$, and the following two equations result:

$$\frac{d^2 h}{dt^2} = -\lambda c^2 h \quad (7.3.7)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi. \quad (7.3.8)$$

We will show that $\lambda > 0$, in which case $h(t)$ is a linear combination of $\sin cv\sqrt{\lambda}t$ and $\cos cv\sqrt{\lambda}t$. The homogeneous boundary conditions imply that the eigenvalue problem is

$$\boxed{\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda \phi} \quad (7.3.9)$$

$$\boxed{\begin{array}{ll} \phi(0, y) = 0 & \phi(x, 0) = 0 \\ \phi(L, y) = 0 & \phi(x, H) = 0; \end{array}} \quad (7.3.10)$$

that is, $\phi = 0$ along the entire boundary. We call (7.3.9)–(7.3.10) a two-dimensional eigenvalue problem.

The eigenvalue problem itself is a linear homogeneous PDE in two independent variables with homogeneous boundary conditions. As such (since the boundaries are simple), we can expect that (7.3.9)–(7.3.10) can be solved by separation of variables in Cartesian coordinates. In other words, we look for product solutions of (7.3.9)–(7.3.10) in the form

$$\phi(x, y) = f(x)g(y). \quad (7.3.11)$$

Before beginning our calculations, let us note that it follows from (7.3.6) that our assumption (7.3.11) is equivalent to

$$u(x, y, t) = f(x)g(y)h(t) \quad (7.3.12)$$

a product of functions of each independent variable. We claim, as we show in an appendix to this section, that we could obtain the same result by substituting (7.3.12) into the wave equation (7.3.1) as we now obtain by substituting (7.3.11) into the two-dimensional eigenvalue problem (7.3.9):

$$g(y) \frac{d^2 f}{dx^2} + f(x) \frac{d^2 g}{dy^2} = -\lambda f(x)g(y). \quad (7.3.13)$$

The x and y parts may be separated by dividing (7.3.13) by $f(x)g(y)$ and rearranging terms:

$$\frac{1}{f} \frac{d^2 f}{dx^2} = -\lambda - \frac{1}{g} \frac{d^2 g}{dy^2} = -\mu. \quad (7.3.14)$$

Since the first expression is only a function of x , while the second is only a function of y we introduce a *second* separation constant. We choose it to be $-\mu$ so that the

easily solved equation, $d^2 f/dx^2 = -\mu f$ has oscillatory solutions (as expected) if $\mu > 0$. Two ordinary differential equations result from separation of variables of a partial differential equation with two independent variables:

$$\boxed{\frac{d^2 f}{dx^2} = -\mu f} \quad (7.3.15)$$

$$\boxed{\frac{d^2 g}{dy^2} = -(\lambda - \mu)g.} \quad (7.3.16)$$

Equations (7.3.15) and (7.3.16) contain *two* separation constants λ and μ , both of which must be determined. In addition, $h(t)$ solves an ordinary differential equation:

$$\boxed{\frac{d^2 h}{dt^2} = -\lambda c^2 h.} \quad (7.3.17)$$

When we separate variables for a partial differential equation in three variables, $u(x, y, t) = f(x)g(y)h(t)$, we obtain three ordinary differential equations, one a function of each independent coordinate. However, there will be only two separation constants.

To determine the separation constants, we need to use the homogeneous boundary conditions (7.3.10). The product form (7.3.11) then implies that

$$\begin{aligned} f(0) &= 0 & \text{and} & & f(L) &= 0 \\ g(0) &= 0 & \text{and} & & g(H) &= 0. \end{aligned} \quad (7.3.18)$$

Of our three ordinary differential equations, only two will be eigenvalue problems. There are homogeneous boundary conditions in x and y . Thus,

$$\frac{d^2 f}{dx^2} = -\mu f \quad \text{with} \quad f(0) = 0 \quad \text{and} \quad f(L) = 0 \quad (7.3.19)$$

is a Sturm-Liouville eigenvalue problem in the x -variable, where μ is the eigenvalue and $f(x)$ is the eigenfunction. Similarly, the y -dependent problem is a regular Sturm-Liouville eigenvalue problem:

$$\frac{d^2 g}{dy^2} = -(\lambda - \mu)g \quad \text{with} \quad g(0) = 0 \quad \text{and} \quad g(H) = 0. \quad (7.3.20)$$

Here λ is the eigenvalue and $g(y)$ the corresponding eigenfunction.

Not only are both (7.3.19) and (7.3.20) Sturm-Liouville eigenvalue problems, but they are both ones we should be quite familiar with. Without going through the well-known details, the eigenvalues are

$$\mu_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 1, 2, 3, \dots, \quad (7.3.21)$$

and the corresponding eigenfunctions are

$$f_n(x) = \sin \frac{n\pi x}{L}. \quad (7.3.22)$$

This determines the allowable values of the separation constant μ_n .

For each value of μ_n , (7.3.20) is still an eigenvalue problem. There is infinite number of eigenvalues λ for each n . Thus, λ should be double subscripted, λ_{nm} . In fact, from (7.3.20) the eigenvalues are

$$\lambda_{nm} - \mu_n = \left(\frac{m\pi}{H}\right)^2, \quad m = 1, 2, 3, \dots, \quad (7.3.23)$$

where we *must* use a different index to represent the various y -eigenvalues (for each value of n). The corresponding y -eigenfunction is

$$g_{nm}(y) = \sin \frac{m\pi y}{H} \quad (7.3.24)$$

The separation constant λ_{nm} now can be determined from (7.3.23):

$$\lambda_{nm} = \mu_n + \left(\frac{m\pi}{H}\right)^2 = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2, \quad (7.3.25)$$

where $n = 1, 2, 3, \dots$ and $m = 1, 2, 3, \dots$. The two-dimensional eigenvalue problem (7.3.9) has eigenvalues λ_{nm} given by (7.3.25) and eigenfunctions given by the product of the two one-dimensional eigenfunctions. Using the notation $\phi_{nm}(x, y)$ for the two-dimensional eigenfunction corresponding to the eigenvalue λ_{nm} , we have

$$\phi_{nm}(x, y) = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}, \quad \begin{array}{l} n = 1, 2, 3, \dots \\ m = 1, 2, 3, \dots \end{array} \quad (7.3.26)$$

Note how easily the homogeneous boundary conditions are satisfied.

From (7.3.25) we have explicitly shown that all the eigenvalues are positive (for this problem). Thus, the time-dependent part of the product solutions are (as previously guessed) $\sin c\sqrt{\lambda_{nm}t}$ and $\cos c\sqrt{\lambda_{nm}t}$, oscillations with natural frequencies $c\sqrt{\lambda_{nm}} = c\sqrt{(n\pi/L)^2 + (m\pi/H)^2}$, $n = 1, 2, 3, \dots$ and $m = 1, 2, 3, \dots$. In considering the displacement u , we have obtained two doubly infinite families of product solutions: $\sin n\pi x/L \sin m\pi y/H \sin c\sqrt{\lambda_{nm}t}$ and $\sin n\pi x/L \sin m\pi y/H \cos c\sqrt{\lambda_{nm}t}$. As with the vibrating string, each of these special product solutions is known as a mode of vibration. We sketch in Fig. 7.3.2 a representation of some of these modes. In each we sketch level contours of displacement in dotted lines at a fixed t . As time varies the shape stays the same, only the amplitude varies periodically. Each mode is a standing wave. Curves along which the displacement is always zero in a mode are called **nodal curves** and are sketched in solid lines. Cells are apparent with neighboring cells always being out of phase; that is, when one cell has a positive

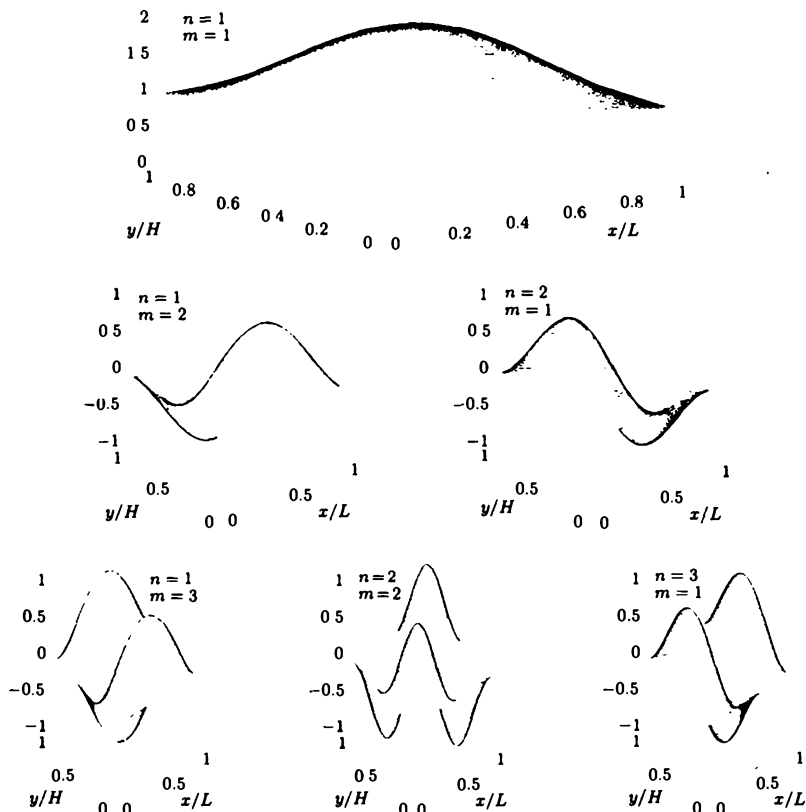


Figure 7.3.2 Nodal curves for modes of a vibrating rectangular membrane.

displacement the neighbor has negative displacement (as represented by the + and - signs).

The principle of superposition implies that we should consider a linear combination of all possible product solutions. Thus, we must include both families, summing over both n and m ,

$$\begin{aligned}
 u(x, y, t) = & \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \cos c\sqrt{\lambda_{nm}}t \\
 & + \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \sin c\sqrt{\lambda_{nm}}t.
 \end{aligned}
 \tag{7.3.27}$$

The two families of coefficients A_{nm} and B_{nm} hopefully will be determined from

the two initial conditions. For example, $u(x, y, 0) = \alpha(x, y)$ implies that

$$\alpha(x, y) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} A_{nm} \sin \frac{n\pi x}{L} \right) \sin \frac{m\pi y}{H}. \quad (7.3.28)$$

The series in (7.3.28) is an example of what is called a **double Fourier series**. Instead of discussing the theory, we show one method to calculate A_{nm} from (7.3.28). (In Sec. 7.4 we will discuss a simpler way.) For fixed x , we note that $\sum_{n=1}^{\infty} A_{nm} \sin n\pi x/L$ depends only on m ; furthermore, it must be the coefficients of the Fourier sine series in y of $\alpha(x, y)$ over $0 < y < H$. From our theory of Fourier sine series, we therefore know that we may easily determine the coefficients:

$$\sum_{n=1}^{\infty} A_{nm} \sin \frac{n\pi x}{L} = \frac{2}{H} \int_0^H \alpha(x, y) \sin \frac{m\pi y}{H} dy, \quad (7.3.29)$$

for each m . Equation (7.3.29) is valid for all x ; the right-hand side is a function of x (not y , because y is integrated from 0 to H). For each m , the left-hand side is a Fourier sine series in x ; in fact, it is the Fourier sine series of the right-hand side, $2/H \int_0^H \alpha(x, y) \sin m\pi y/H dy$. The coefficients of this Fourier sine series in x are easily determined:

$$A_{nm} = \frac{2}{L} \int_0^L \left[\frac{2}{H} \int_0^H \alpha(x, y) \sin \frac{m\pi y}{H} dy \right] \sin \frac{n\pi x}{L} dx. \quad (7.3.30)$$

This may be simplified to one double integral over the entire rectangular region, rather than two iterated one-dimensional integrals. In this manner we have determined one set of coefficients from one of the initial conditions.

The other coefficients B_{nm} can be determined in a similar way. In particular, from (7.3.27), $\partial u/\partial t(x, y, 0) = \beta(x, y)$, which implies that

$$\beta(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c\sqrt{\lambda_{nm}} B_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \quad (7.3.31)$$

Thus, again using a Fourier sine series in y and a Fourier sine series in x , we obtain

$$c\sqrt{\lambda_{nm}} B_{nm} = \frac{4}{LH} \int_0^L \int_0^H \beta(x, y) \sin \frac{m\pi y}{H} \sin \frac{n\pi x}{L} dy dx. \quad (7.3.32)$$

The solution of our initial value problem is the doubly infinite series given by (7.3.27), where the coefficients are determined by (7.3.30) and (7.3.32).

We have shown that when all three independent variables separate for a partial differential equation, there results three ordinary differential equations, two of which

are eigenvalue problems. In general, for a partial differential equation in N variables that completely separates, there will be N ordinary differential equations, $N - 1$ of which are one-dimensional eigenvalue problems (to determine the $N - 1$ separation constants). We have already shown this for $N = 3$ (this section) and $N = 2$.

EXERCISES 7.3

7.3.1. Consider the heat equation in a two-dimensional rectangular region $0 < x < L, 0 < y < H$,

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial condition

$$u(x, y, 0) = f(x, y).$$

Solve the initial value problem and analyze the temperature as $t \rightarrow \infty$ if the boundary conditions are

- * (a) $u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, H, t) = 0$
- (b) $\frac{\partial u}{\partial x}(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0$
- * (c) $\frac{\partial u}{\partial x}(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad u(x, 0, t) = 0, \quad u(x, H, t) = 0$
- (d) $u(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0$
- (e) $u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad u(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) + hu(x, H, t) = 0. \quad (h > 0)$

7.3.2. Consider the heat equation in a three-dimensional box-shaped region, $0 < x < L, 0 < y < H, 0 < z < W$,

$$\frac{\partial u}{\partial t} = k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

subject to the initial condition

$$u(x, y, z, 0) = f(x, y, z).$$

Solve the initial value problem and analyze the temperature as $t \rightarrow \infty$ if the boundary conditions are

- (a) $u(0, y, z, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, z, t) = 0, \quad \frac{\partial u}{\partial z}(x, y, 0, t) = 0,$
 $u(L, y, z, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, z, t) = 0, \quad u(x, y, W, t) = 0$
- * (b) $\frac{\partial u}{\partial x}(0, y, z, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, z, t) = 0, \quad \frac{\partial u}{\partial z}(x, y, 0, t) = 0,$
 $\frac{\partial u}{\partial x}(L, y, z, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, z, t) = 0, \quad \frac{\partial u}{\partial z}(x, y, W, t) = 0$

7.3.3 Solve

$$\frac{\partial u}{\partial t} = k_1 \frac{\partial^2 u}{\partial x^2} + k_2 \frac{\partial^2 u}{\partial y^2}$$

on a rectangle ($0 < x < L, 0 < y < H$) subject to

$$u(x, y, 0) = f(x, y) \quad \begin{array}{l} u(0, y, t) = 0 \\ u(L, y, t) = 0 \end{array} \quad \begin{array}{l} \frac{\partial u}{\partial y}(x, 0, t) = 0 \\ \frac{\partial u}{\partial y}(x, H, t) = 0. \end{array}$$

7.3.4. Consider the wave equation for a vibrating rectangular membrane ($0 < x < L, 0 < y < H$)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

subject to the initial conditions

$$u(x, y, 0) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(x, y, 0) = f(x, y).$$

Solve the initial value problem if

$$(a) \quad u(0, y, t) = 0, \quad u(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0$$

$$* (b) \quad \frac{\partial u}{\partial x}(0, y, t) = 0, \quad \frac{\partial u}{\partial x}(L, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, H, t) = 0$$

7.3.5. Consider

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - k \frac{\partial u}{\partial t} \quad \text{with } k > 0.$$

(a) Give a *brief* physical interpretation of this equation.(b) Suppose that $u(x, y, t) = f(x)g(y)h(t)$. What ordinary differential equations are satisfied by f , g , and h ?

7.3.6. Consider Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

in a right cylinder whose base is arbitrarily shaped (see Fig. 7.3.3). The top is $z = H$ and the bottom is $z = 0$. Assume that

$$\begin{array}{l} \frac{\partial}{\partial z} u(x, y, 0) = 0 \\ u(x, y, H) = f(x, y) \end{array}$$

and $u = 0$ on the "lateral" sides.(a) Separate the z -variable in general.*(b) Solve for $u(x, y, z)$ if the region is a rectangular box, $0 < x < L, 0 < y < W, 0 < z < H$.

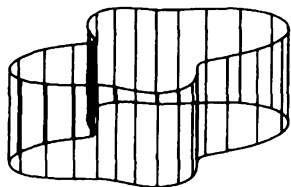


Figure 7.3.3

7.3.7. If possible, solve Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

in a rectangular-shaped region, $0 < x < L, 0 < y < W, 0 < z < H$, subject to the boundary conditions

(a) $\frac{\partial u}{\partial x}(0, y, z) = 0,$	$u(x, 0, z) = 0,$	$u(x, y, 0) = f(x, y)$
$\frac{\partial u}{\partial x}(L, y, z) = 0,$	$u(x, W, z) = 0,$	$u(x, y, H) = 0$
(b) $u(0, y, z) = 0,$	$u(x, 0, z) = 0,$	$u(x, y, 0) = 0,$
$u(L, y, z) = 0,$	$u(x, W, z) = f(x, z),$	$u(x, y, H) = 0$
* (c) $\frac{\partial u}{\partial x}(0, y, z) = 0,$	$\frac{\partial u}{\partial y}(x, 0, z) = 0,$	$\frac{\partial u}{\partial z}(x, y, 0) = 0$
$\frac{\partial u}{\partial x}(L, y, z) = f(y, z),$	$\frac{\partial u}{\partial y}(x, W, z) = 0,$	$\frac{\partial u}{\partial z}(x, y, H) = 0$
* (d) $\frac{\partial u}{\partial x}(0, y, z) = 0,$	$\frac{\partial u}{\partial y}(x, 0, z) = 0,$	$\frac{\partial u}{\partial z}(x, y, 0) = 0$
$u(L, y, z) = g(y, z),$	$\frac{\partial u}{\partial y}(x, W, z) = 0,$	$\frac{\partial u}{\partial z}(x, y, H) = 0$

Appendix to 7.3: Outline of Alternative Method to Separate Variables

An alternative (and equivalent) method to separate variables for

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (7.3.33)$$

is to assume product solutions of the form

$$u(x, y, t) = f(x)g(y)h(t). \quad (7.3.34)$$

By substituting (7.3.34) into (7.3.33) and dividing by $c^2 f(x)g(y)h(t)$, we obtain

$$\frac{1}{c^2} \frac{1}{h} \frac{d^2 h}{dt^2} = \frac{1}{f} \frac{d^2 f}{dx^2} + \frac{1}{g} \frac{d^2 g}{dy^2} = -\lambda, \quad (7.3.35)$$

after introducing a separation constant $-\lambda$. This shows that

$$\boxed{\frac{d^2h}{dt^2} = -\lambda c^2 h.} \quad (7.3.36)$$

Equation (7.3.35) can be separated further,

$$\frac{1}{f} \frac{d^2 f}{dx^2} = -\lambda - \frac{1}{g} \frac{d^2 g}{dy^2} = -\mu, \quad (7.3.37)$$

enabling a second separation constant $-\mu$ to be introduced:

$$\boxed{\frac{d^2 f}{dx^2} = -\mu f} \quad (7.3.38)$$

$$\boxed{\frac{d^2 g}{dy^2} = -(\lambda - \mu)g.} \quad (7.3.39)$$

In this way we have derived the same three ordinary differential equations (with two separation constants).

7.4 Statements and Illustrations of Theorems for the Eigenvalue Problem $\nabla^2\phi + \lambda\phi = 0$

In solving the heat equation and the wave equation in any two- or three-dimensional region R (with constant physical properties, such as density), we have shown that the spatial part $\phi(x, y, z)$ of product form solutions $u(x, y, z, t) = \phi(x, y, z)h(t)$ satisfies the following multidimensional eigenvalue problem:

$$\boxed{\nabla^2\phi + \lambda\phi = 0,} \quad (7.4.1)$$

with

$$a\phi + b\nabla\phi \cdot \hat{n} = 0 \quad (7.4.2)$$

on the entire boundary. Here a and b can depend on x , y , and z . Equation (7.4.1) is known as the **Helmholtz equation**.

Equation (7.4.1) can be generalized to

$$\nabla \cdot (p\nabla\phi) + q\phi + \lambda\sigma\phi = 0, \quad (7.4.3)$$

where p , q and σ are functions of x , y , and z . This eigenvalue problem [with boundary condition (7.4.2)] is directly analogous to the one-dimensional regular Sturm-Liouville eigenvalue problem. We prefer to deal with a somewhat simpler case, (7.4.1), corresponding to $p = \sigma = 1$ and $q = 0$. We will state and prove results for (7.4.1). We leave the discussion of (7.4.3) to some exercises (in Sec. 7.5).

Only for very simple geometries (for example, rectangles, see Sec. 7.3, or circles, see Sec. 7.7) can (7.4.1) be solved explicitly. In other situations, we may have to rely on numerical treatments. However, certain general properties of (7.4.1) are quite useful, all analogous to results we understand for the one-dimensional Sturm-Liouville problem. The reasons for the analogy will be discussed in the next section. We begin by simply stating the theorems for the two-dimensional case of (7.4.1) and (7.4.2):

1. All the eigenvalues are real.
2. There exists an infinite number of eigenvalues. There is a smallest eigenvalue, but no largest one.
3. Corresponding to an eigenvalue, there *may* be many eigenfunctions (unlike regular Sturm-Liouville eigenvalue problems).
4. The eigenfunctions $\phi(x, y)$ form a "complete" set, meaning that any piecewise smooth function $f(x, y)$ can be represented by a generalized Fourier series of the eigenfunctions:

$$f(x, y) \sim \sum_{\lambda} a_{\lambda} \phi_{\lambda}(x, y). \quad (7.4.4)$$

Here $\sum_{\lambda} a_{\lambda} \phi_{\lambda}$ means a linear combination of all the eigenfunctions. The series converges in the mean if the coefficients a_{λ} are chosen correctly.

5. Eigenfunctions belonging to different eigenvalues (λ_1 and λ_2) are orthogonal relative to the weight σ ($\sigma = 1$) over the entire region R . Mathematically,

$$\iint_R \phi_{\lambda_1} \phi_{\lambda_2} dx dy = 0 \quad \text{if } \lambda_1 \neq \lambda_2, \quad (7.4.5)$$

where $\iint_R dx dy$ represents an integral over the entire region R . Furthermore, different eigenfunctions belonging to the same eigenvalue can be made orthogonal by the Gram-Schmidt process (see Sec. 7.5). Thus, we may assume that (7.4.5) is valid even if $\lambda_1 = \lambda_2$ as long as ϕ_{λ_1} is independent of ϕ_{λ_2} .

6. An eigenvalue λ can be related to the eigenfunction by the Rayleigh quotient:

$$\lambda = \frac{-\oint \phi \nabla \phi \cdot \hat{n} dx + \iint_R |\nabla \phi|^2 dx dy}{\iint_R \phi^2 dx dy}. \quad (7.4.6)$$

The boundary conditions often simplify the boundary integral.

Here \hat{n} is a unit outward normal and $\oint ds$ is a closed line integral over the entire boundary of the plane two-dimensional region, where ds is the differential arc length. The three-dimensional result is nearly identical; \iint must be replaced by \iiint and the boundary line integral $\oint ds$ must be replaced by the boundary surface integral $\oint dS$, where dS is the differential surface area.

Example. We will prove some of these statements in Sec. 7.5. To understand their meaning, we will show how the example of Sec. 7.3 illustrates most of these theorems. For the vibrations of a rectangular ($0 < x < L$, $0 < y < H$) membrane with fixed zero boundary conditions, we have shown that the relevant eigenvalue problem is

$$\begin{aligned} \nabla^2\phi + \lambda\phi &= 0 \\ \phi(0, y) = 0 \quad \phi(x, 0) &= 0 \\ \phi(L, y) = 0 \quad \phi(x, H) &= 0. \end{aligned} \tag{7.4.7}$$

We have determined that the eigenvalues and corresponding eigenfunctions are

$$\lambda_{nm} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2 \quad \begin{matrix} n = 1, 2, 3, \dots \\ m = 1, 2, 3, \dots \end{matrix} \quad \text{with} \quad \phi_{nm}(x, y) = \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{7.4.8}$$

- 1. Real eigenvalues.** In our calculation of the eigenvalues for (7.4.7) we assumed that the eigenfunctions existed in a product form. Under that assumption, (7.4.8) showed the eigenvalues to be real. Our theorem guarantees that the eigenvalues will always be real.
- 2. Ordering of eigenvalues.** There is a doubly infinite set of eigenvalues for (7.4.7), namely $\lambda_{nm} = (n\pi/L)^2 + (m\pi/H)^2$ for $n = 1, 2, 3, \dots$ and $m = 1, 2, 3, \dots$. There is a smallest eigenvalue, $\lambda_{11} = (\pi/L)^2 + (\pi/H)^2$, but no largest eigenvalue.
- 3. Multiple eigenvalues.** For $\nabla^2\phi + \lambda\phi = 0$, our theorem states that, in general, it is possible for there to be more than one eigenfunction corresponding to the same eigenvalue. To illustrate this, consider (7.4.7) in the case in which $L = 2H$. Then

$$\lambda_{nm} = \frac{\pi^2}{4H^2} (n^2 + 4m^2) \tag{7.4.9}$$

with

$$\phi_{nm} = \sin \frac{n\pi x}{2H} \sin \frac{m\pi y}{H}. \tag{7.4.10}$$

We note that it is possible to have different eigenfunctions corresponding to the same eigenvalue. For example, $n = 4$, $m = 1$ and $n = 2$, $m = 2$, yield the same eigenvalue:

$$\lambda_{41} = \lambda_{22} = \frac{\pi^2}{4H^2} 20.$$

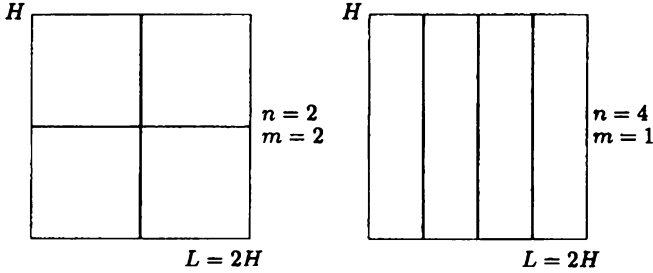


Figure 7.4.1 Nodal curves for eigenfunctions with the same eigenvalue (symmetric).

For $n = 4, m = 1$, the eigenfunction is $\phi_{41} = \sin 4\pi x/2H \sin \pi y/H$, while for $n = 2, m = 2$, $\phi_{22} = \sin 2\pi x/2H \sin 2\pi y/H$. The nodal curves for these eigenfunctions are sketched in Fig. 7.4.1. They are different eigenfunctions with the same eigenvalue, $\lambda = (\pi^2/4H^2)20$. It is not surprising that the eigenvalue is the same, since a membrane vibrating in these modes has cells of the same dimensions: one $H \times H/2$ and the other $H/2 \times H$. By symmetry they will have the same natural frequency (and hence the same eigenvalue since the natural frequency is $c\sqrt{\lambda}$). In fact, in general by symmetry [as well as by formula (7.4.9)] $\lambda_{(2n)m} = \lambda_{(2m)n}$.

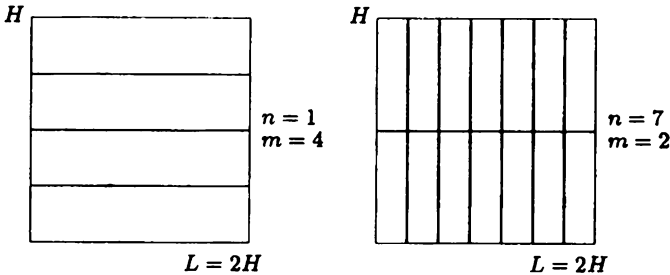


Figure 7.4.2 Nodal curves for eigenfunction with the same eigenvalue (asymmetric).

However, it is also possible for more than one eigenfunction to occur for reasons having nothing to do with symmetry. For example, $n = 1, m = 4$ and $n = 7, m = 2$ yield the same eigenvalue: $\lambda_{14} = \lambda_{72} = (\pi^2/4H^2)65$. The corresponding eigenfunctions are $\phi_{14} = \sin \pi x/2H \sin 4\pi y/H$ and $\phi_{72} = \sin 7\pi x/2H \sin 2\pi y/H$, which are sketched in Fig. 7.4.2. It is only coincidental that both of these shapes vibrate with the same frequency. In these situations, it is possible for two eigenfunctions to correspond to the same eigenvalue. We can find situations with even more multiplicities (or degeneracies). Since $\lambda_{14} = \lambda_{72} = (\pi^2/4H^2)65$, it is also true that $\lambda_{28} = \lambda_{(14)4} = (\pi^2/4H^2)260$.

However, by symmetry $\lambda_{28} = \lambda_{(16)1}$ and $\lambda_{(14)4} = \lambda_{87}$. Thus,

$$\lambda_{28} = \lambda_{(16)1} = \lambda_{(14)4} = \lambda_{87} = \left(\frac{\pi^2}{4H^2}\right) 260.$$

Here there are four eigenfunctions corresponding to the same eigenvalue.

4a. **Series of eigenfunctions.** According to this theorem, (7.4.4), the eigenfunctions of $\nabla^2 \phi + \lambda \phi = 0$ can always be used to represent any piecewise smooth function $f(x, y)$. In our illustrative example, (7.4.7), \sum_{λ} becomes a double sum,

$$f(x, y) \sim \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_{nm} \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H}. \tag{7.4.11}$$

5. **Orthogonality of eigenfunctions.** We will show that the multidimensional orthogonality of the eigenfunctions, as expressed by (7.4.5) for any two different eigenfunctions, can be used to determine the generalized Fourier coefficients in (7.4.4).¹ We will do this in exactly the way we did for one-dimensional Sturm-Liouville eigenfunction expansions. We simply multiply (7.4.4) by ϕ_{λ} , and integrate over the entire region R :

$$\iint_R f \phi_{\lambda} \, dx \, dy = \sum_{\lambda} a_{\lambda} \iint_R \phi_{\lambda} \phi_{\lambda} \, dx \, dy. \tag{7.4.12}$$

Since the eigenfunctions are all orthogonal to each other (with weight 1 because $\nabla^2 \phi + \lambda \phi = 0$), it follows that

$$\iint_R f \phi_{\lambda} \, dx \, dy = a_{\lambda} \iint_R \phi_{\lambda}^2 \, dx \, dy, \tag{7.4.13}$$

or, equivalently,

$$a_{\lambda} = \frac{\iint_R f \phi_{\lambda} \, dx \, dy}{\iint_R \phi_{\lambda}^2 \, dx \, dy}. \tag{7.4.14}$$

There is no difficulty in forming (7.4.14) from (7.4.13) since the denominator of (7.4.14) is necessarily positive.

For the special case that occurs for a rectangle with fixed zero boundary conditions, (7.4.7), the generalized Fourier coefficients a_{nm} are given by (7.4.14):

$$a_{nm} = \frac{\int_0^H \int_0^L f(x, y) \sin \frac{n\pi x}{L} \sin \frac{m\pi y}{H} \, dx \, dy}{\int_0^H \int_0^L \sin^2 \frac{n\pi x}{L} \sin^2 \frac{m\pi y}{H} \, dx \, dy}. \tag{7.4.15}$$

The integral in the denominator may be easily shown to equal $(L/2)(H/2)$ by calculating two one-dimensional integrals; in this way we rederive (7.3.30). In

¹If there is more than one eigenfunction corresponding to the same eigenvalue, then we assume that the eigenfunctions have been made orthogonal (if necessary by the Gram-Schmidt process).

this case, (7.4.11), the generalized Fourier coefficient a_{nm} can be evaluated in two equivalent ways:

- (a) Using one two-dimensional orthogonality formula for the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$
- (b) Using two one-dimensional orthogonality formulas

4b. **Convergence.** As with any Sturm-Liouville eigenvalue problem (see Sec. 5.10), a *finite* series of the eigenfunctions of $\nabla^2\phi + \lambda\phi = 0$ may be used to approximate a function $f(x, y)$. In particular, we could show that if we measure error in the mean-square sense,

$$E \equiv \iint_R \left(f - \sum_{\lambda} a_{\lambda} \phi_{\lambda} \right)^2 dx dy, \quad (7.4.16)$$

with weight function 1, then this mean-square error is minimized by the coefficients a_{λ} being chosen by (7.4.14), the generalized Fourier coefficients. It is known that the approximation improves as the number of terms increases. Furthermore, $E \rightarrow 0$ as all the eigenfunctions are included. We say that the series $\sum_{\lambda} a_{\lambda} \phi_{\lambda}$ **converges in the mean** to f .

EXERCISES 7.4

7.4.1. Consider the eigenvalue problem

$$\begin{aligned} \nabla^2\phi + \lambda\phi &= 0 \\ \frac{\partial\phi}{\partial x}(0, y) &= 0 & \phi(x, 0) &= 0 \\ \frac{\partial\phi}{\partial x}(L, y) &= 0 & \phi(x, H) &= 0. \end{aligned}$$

- *(a) Show that there is a doubly infinite set of eigenvalues.
- (b) If $L = H$, show that most eigenvalues have more than one eigenfunction.
- (c) Derive that the eigenfunctions are orthogonal in a two-dimensional sense using two one-dimensional orthogonality relations.

7.4.2. Without using the explicit solution of (7.4.7), show that $\lambda \geq 0$ from the Rayleigh quotient, (7.4.6).

7.4.3. If necessary, see Sec. 7.5:

- (a) Derive that $\iint (u\nabla^2v - v\nabla^2u) dx dy = \oint (u\nabla v - v\nabla u) \cdot \hat{n} ds$.
- (b) From part (a), derive (7.4.5).

7.4.4. Derive (7.4.6). If necessary, see Sec. 7.6. [*Hint:* Multiply (7.4.1) by ϕ and integrate.]